1. The prediction limits can be used in conjunction with a normalizing transformation, as illustrated by the logarithmic transformation in the preceding example from Nelson (1982). Transformations to achieve approximate normality are especially important in light of the lack of robustness of the prediction limits noted earlier.

2. Some instructors may wish to contrast prediction limits with tolerance limits; the former refers to limits for a single independent observation from a normal population, and the latter refers to limits within which a certain fraction of the entire normal population is claimed to lie.

3. The prediction limits (2.3) can be extended readily to include the case of predicting the mean of m new observations in an independent random sample from the same normal population. In this case the $1 - \alpha$ limits for the predicted mean have the form

$$\bar{Y} \pm t_{1-\alpha/2,n-1} s\left[\frac{1}{m} + \frac{1}{n}\right]^{1/2}. \qquad (3.1)$$

In addition, prediction limits of the form $\bar{Y} \pm rs$ can be constructed that will contain *all m* new observations with a given level of confidence. Tables of the multiplier r may be found in Hahn (1969, 1970).

4. The instructor may wish to note that result (2.2) holds approximately for the standardized sample values

$$Z_i = (Y_i - \bar{Y})/s, \qquad i = 1, \ldots, n, \qquad (3.2)$$

and this fact may be useful for outlier identification in normal samples. The class discussion of (3.2) provides a useful preparation for the subsequent discussion of standardized residuals in a regression framework.

5. For courses with a Bayesian orientation, the limits in (2.3) are the central $1 - \alpha$ posterior probability limits for a prediction of Y based on a normal sample $Y_1, \ldots, Y_n$ and an uninformative improper prior joint density function for μ and σ (e.g., see DeGroot 1970, chap. 10).

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Relationships Among Common Univariate Distributions

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Common univariate distributions are usually discussed separately in introductory probability textbooks, which makes it difficult for students to understand the relationships among these distributions. The purpose of this article is to present a figure that illustrates some of these relationships.

KEY WORDS: Limiting distributions; Transformations of random variables.

1. INTRODUCTION

Students in a first course in probability usually study common univariate distributions. Most introductory textbooks discuss each of the distributions in separate sections. One of the drawbacks of this approach is that students often do not grasp all of the interrelationships among the distributions. The purpose of this article is to present and discuss a figure that overcomes this shortfall.

There are several excellent sources for studying univariate distributions. Hastings and Peacock's (1975) handbook shows graphs of densities and variate relationships for several distributions. Hirano, Kuboki, Aki, and Kuribayashi (1983) gave graphs of univariate distributions for many combinations of parameter values. For more detail, Johnson and Kotz (1970) have done a four-volume series covering univariate and multivariate distributions. Recently, Patil, Boswell, Joshi, and Ratnaparkhi (1985) and Patil, Boswell, and Ratnaparkhi (1985) have also completed volumes on discrete and continuous distributions. Other books on distributions and modeling include Ord (1972), Patel, Kapadia, and Owen (1976), and Shapiro and Gross (1981). Diagrams that relate these distributions to one another may be found in Nakagawa and Yoda (1977), Taha (1982), and Marshall and Olkin (1985).

2. DISCUSSION

The diagram in Figure 1 shows some relationships among common univariate distributions that might be presented in

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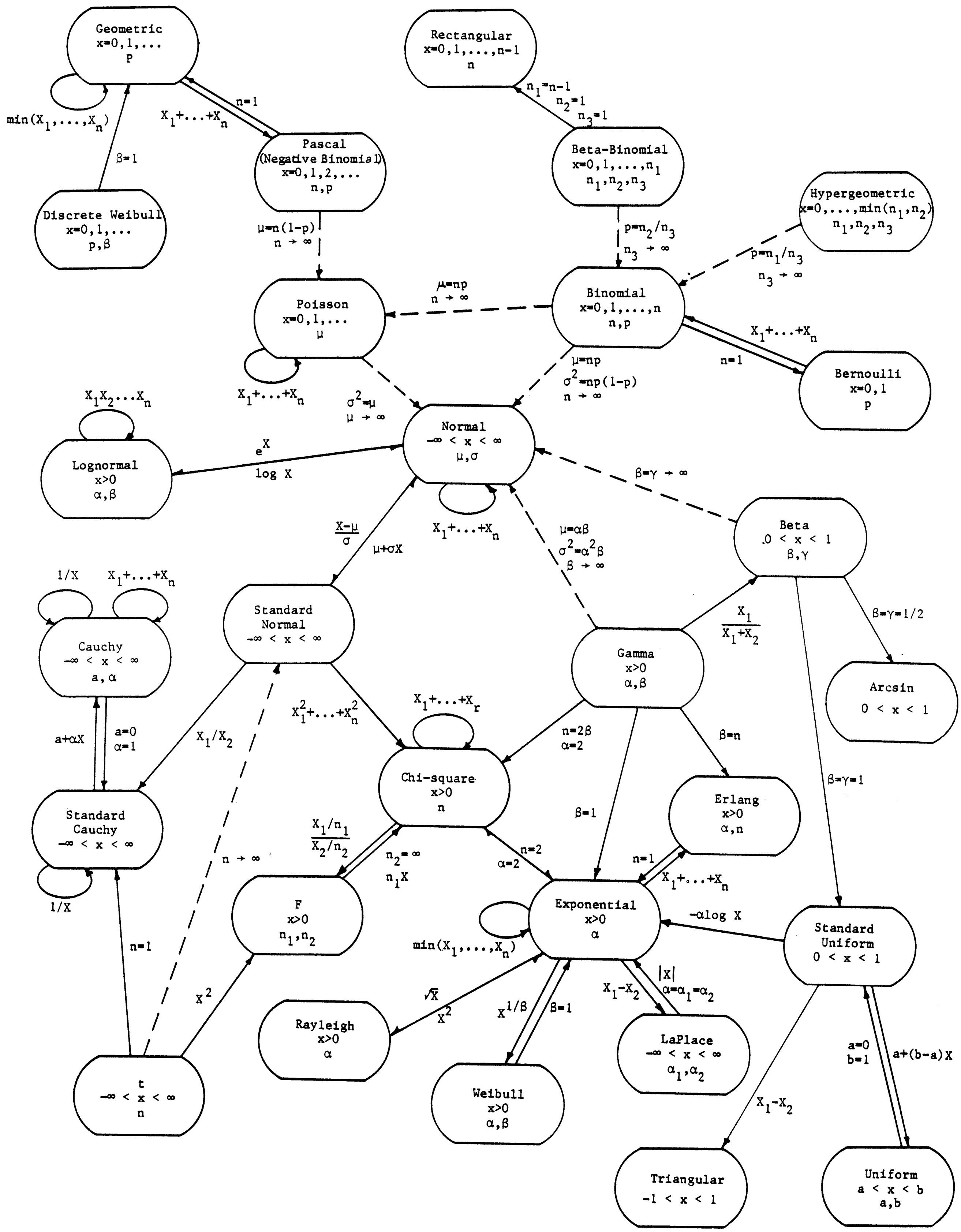

Figure 1. Relationships Among Distributions.

an introductory probability course. There are 9 discrete distributions, shown on the upper part of the diagram, and 19 continuous distributions. The first line of each entry is the name of the distribution. The next line contains the region of support for the distribution. The last line contains the distribution's parameters. The parameters must satisfy the following: n is an integer; $0 < p < 1$; α and σ are positive scale parameters; β and γ are positive shape parameters; and μ, a, and b are location parameters. When μ or σ are used as parameters, they denote the mean and standard deviation of the distribution, respectively.

There are three types of relationships among distributions: limiting distributions, transformations, and special cases. Limiting distributions are indicated with a dashed arrow. Transformations (which assume independent random variables) and special cases are indicated by a solid arrow. One-to-one transformations have an arrow pointing in both directions. The random variable X is used for all distributions. Thus the arrow from the t distribution to the F distribution indicates that the square of a t random variable has an F distribution.

The normal and exponential distributions play a central role in Figure 1. This fact is partially due to their natural genesis via the central limit theorem and superpositioning principle. In addition, some distributions generalize the exponential distribution (such as the Weibull), since it is often used in reliability to model component lifetimes.

The transformation relationships in Figure 1 can be combined to form other relationships. A path from the standard normal to the chi-square to the exponential to the Rayleigh, for example, indicates that the random variable $(X_1^2 + X_2^2)^{1/2}$ has the Rayleigh distribution if X_1 and X_2 are standard normal random variables.

The relationship between the uniform (0, 1) distribution and the exponential distribution is valid by the probability integral transformation. Since the probability integral transformation states that the cumulative distribution function for a random variable is uniformly distributed between 0 and 1, an arrow could be drawn from the uniform (0, 1) distribution to every other distribution shown in the figure, although not all of these relationships are closed form. This relationship is known in the simulation literature as the inverse-cdf technique for random variate generation. A survey article of general methods for random variate generation is given by Schmeiser (1980).

The probability mass functions and probability density functions for the distributions in Figure 1 are listed in the Appendix. All continuous lifetime distributions [i.e., those with support on $(0, \infty)$] may be generalized to have support on (a, ∞) by replacing x with $x - a$ in the density function. In addition, the parameterizations chosen for the distributions in Figure 1 are not unique.

There are many relationships that Figure 1 does not indicate. First, because of space constraints, there are relationships (e.g., between the exponential and Poisson distributions) that are not included. Second, combining two random variables with different distributions is not included. For example, the defining formula for the Student-t distribution, $Z(\chi^2/n)^{1/2}$, where Z is standard normal and χ^2 has the chi-square distribution with n df, is not shown. Third, analogies between discrete and continuous distributions (e.g., the geometric and exponential) are not shown. Finally, the distributions included are oriented toward the classical distributions, and families of distributions (e.g., the Pearson system) are not included.

3. CONCLUSIONS

There are two applications for this diagram. First, after presenting common univariate distributions in an introductory course in probability, it can be used to indicate how distributions relate to one another. Second, in an advanced course (e.g., simulation or reliability), Figure 1 provides a quick review of important univariate distributions.

APPENDIX: DISTRIBUTION PARAMETERIZATIONS

Discrete Distributions

Bernoulli:

$$f(x) = p^x(1 - p)^{1-x}, \qquad x = 0, 1$$

Beta-Binomial:

$$f(x) = \binom{n_2 + x - 1}{x} \times \binom{n_1 + n_3 - x - 1}{n_1 - x} \Big/ \binom{n_1 + n_2 + n_3 - 1}{n_1}, \qquad x = 0, 1, \ldots, n_1$$

Binomial:

$$f(x) = \binom{n}{x} p^x(1 - p)^{n-x}, \qquad x = 0, 1, \ldots, n$$

Discrete Weibull:

$$f(x) = (1 - p)^{x^\beta} - (1 - p)^{(x+1)^\beta}, \qquad x = 0, 1, \ldots$$

Geometric:

$$f(x) = p(1 - p)^x, \qquad x = 0, 1, \ldots$$

Hypergeometric:

$$f(x) = \binom{n_1}{x}\binom{n_3 - n_1}{n_2 - x} \Big/ \binom{n_3}{n_2}, \qquad x = 0, 1, \ldots, \min(n_1, n_2)$$

Pascal (negative binomial):

$$f(x) = \binom{n - 1 + x}{x} p^n (1 - p)^x, \qquad x = 0, 1, \ldots$$

Poisson:

$$f(x) = \mu^x e^{-\mu}/x!, \qquad x = 0, 1, \ldots$$

Rectangular:

$$f(x) = 1/n, \qquad x = 0, 1, \ldots, n - 1$$

Continuous Distributions

Arcsin:

$$f(x) = 1/\pi[x(1 - x)]^{1/2}, \qquad 0 < x < 1$$

Beta:
$$f(x) = [\Gamma(\beta + \gamma)/\Gamma(\beta)\Gamma(\gamma)] \times x^{\beta-1}(1 - x)^{\gamma-1}, \qquad 0 < x < 1$$

Cauchy:
$$f(x) = 1/\alpha\pi[1 + ((x - a)/\alpha)^2], \qquad -\infty < x < \infty$$

Chi-square:
$$f(x) = [1/2^{n/2}\Gamma(n/2)]\, x^{n/2-1}e^{-x/2}, \qquad x > 0$$

Erlang:
$$f(x) = [1/\alpha^n(n - 1)!]\, x^{n-1}e^{-x/\alpha}, \qquad x > 0$$

Exponential:
$$f(x) = (1/\alpha)\, e^{-x/\alpha}, \qquad x > 0$$

F:
$$f(x) = \Gamma((n_1 + n_2)/2) \times (n_1/n_2)^{n_1/2}\, x^{n_1/2-1}/ \Gamma(n_1/2)\,\Gamma(n_2/2) \times [(n_1/n_2)\, x + 1]^{((n_1 + n_2)/2)}, \qquad x > 0$$

Gamma:
$$f(x) = [1/\alpha^\beta\Gamma(\beta)]\, x^{\beta-1}e^{-x/\alpha}, \qquad x > 0$$

LaPlace:
$$f(x) = [1/(\alpha_1 + \alpha_2)]e^{-x/\alpha_1}, \qquad x \geq 0$$
$$= [1/(\alpha_1 + \alpha_2)]e^{x/\alpha_2}, \qquad x < 0$$

Lognormal:
$$f(x) = [1/(2\pi)^{1/2}\, x\beta]\exp[-½(\log(x/\alpha)/\beta)^2], \qquad x > 0$$

Normal:
$$f(x) = [1/(2\pi)^{1/2}\sigma]\exp[-½((x - \mu)/\sigma)^2], \qquad -\infty < x < \infty$$

Rayleigh:
$$f(x) = (2x/\alpha)\, e^{-x^2/\alpha}, \qquad x > 0$$

t:
$$f(x) = \Gamma((n + 1)/2)/(n\pi)^{1/2}\Gamma(n/2) \times [x^2/n + 1]^{(n+1)/2}, \qquad -\infty < x < \infty$$

Triangular:
$$f(x) = 1 + x, \qquad -1 < x < 0$$
$$= 1 - x, \qquad 0 \leq x < 1$$

Uniform:
$$f(x) = 1/(b - a), \qquad a < x < b$$

Weibull:
$$f(x) = (1/\alpha)\, \beta x^{\beta-1}\exp[-(1/\alpha)x^\beta], \qquad x > 0$$

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