Quasi-likelihood Estimation on SFIEGARCH Process: a Simulation Study

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Abstract

The theoretical properties of the quasi-maximum likelihood estimate (QMLE) for SFIEGARCH models are still an open issue. This work presents a simulation study to access the finite sample performance of this estimator for misspecified SFIEGARCH$(0,d,0)_s$ process. The simulation study presented here considers the Student's t and the generalized error distribution (GED) as underlying distributions for the innovation process. In order to obtain lighter and heavier tails than the Gaussian distribution, different values for the nuisance parameter ν are considered for both underlying distributions.

Keywords: SFIEGARCH Process; QMLE Method; Misspecification.

1 Introduction

In the last years economists have noticed that volatility of high frequency financial time series shows long range dependence merged with periodic behavior. According to Bordignon et al. (2009), in the case of exchange rate returns, this pattern is generally attributed to different openings of European, Asian and North American markets superimposed each other. Stock markets present a similar pattern, mainly due to the so-called time-of-day phenomena, such as market opening, closing operations, lunch-hour and overlapping effects.

To account for the long memory periodic behavior observed in financial time series, Bordignon et al. (2009) and Lopes and Prass (2012) introduce, respectively, the PLM-EGARCH and the SFIEGARCH processes. Both processes are extensions of the well known FIEGARCH process (Nelson, 1991). A discussion on their similarities and differences can be found in Lopes and Prass (2012).

This work presents a simulation study to access the finite sample performance of quasi-maximum likelihood estimate (QMLE) for SFIEGARCH$(0,d,0)_s$ processes

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when the underlying distribution functions are misspecified. The asymptotic properties for the quasi-likelihood method are well established for ARCH/GARCH models (see, for instance Lee and Hansen, 1994; Lumsdaine, 1996; Berkes et al., 2003; Berkes and Horváth, 2003; Hall and Yao, 2003) and also for EGARCH models (see, for instance Straumann and Mikosch, 2006). However, at our best knowledge, there are no theoretical results regarding SFIEGARCH processes in the literature.

The paper is organized as follows. Section 2 introduces SFIEGARCH processes, presents some results regarding stationarity and ergodicity of these processes and gives a recurrence formula to calculate the coefficients for the polynomial needed to generate samples from these processes. Section 3 describes the quasi-maximum likelihood procedure. Section 4 describes the simulation study and results. Section 5 concludes the paper.

2 SFIEGARCH Process

Let $d \in \mathbb{R}$ and $(1-\mathcal{B}^s)^{-d}$ be the operator defined by its Maclaurin series expansion, namely,

$$(1-\mathcal{B}^s)^{-d} = \sum_{k=0}^{\infty} \frac{\Gamma(k+d)}{\Gamma(k+1)\Gamma(d)} (\mathcal{B}^s)^k := \sum_{k=0}^{\infty} \delta_{-d,k} \mathcal{B}^{sk} := \sum_{k=0}^{\infty} \pi_{d,k} \mathcal{B}^k, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function, $\mathcal{B}$ is the backward shift operator defined by $\mathcal{B}^{sk}(X_t) = X_{t-sk}$, for all $t \in \mathbb{Z}$ and $s, k \in \mathbb{N}$, $\pi_{d,sj+r} := 0$, for all $r \in \{1, \cdots, s-1\}$, and $\pi_{d,sj} := \delta_{-d,j} := \Gamma(j+d)\big[\Gamma(j+1)\Gamma(d)\big]^{-1}$, for all $j \in \mathbb{N}$.

Consider the order p and q polynomials, respectively, defined by

$$\alpha(z) := \sum_{i=0}^{p} (-\alpha_i) z^i \quad \text{and} \quad \beta(z) := \sum_{j=0}^{q} (-\beta_j) z^j, \quad \text{with} \quad \alpha_0 = \beta_0 = -1.$$

Assuming that $\alpha(\cdot)$ and $\beta(\cdot)$ have no common roots and that $\beta(z) \neq 0$ in the closed disk $\{z : |z| \leq 1\}$, define $\lambda(\cdot)$ by

$$\lambda(z) := \frac{\alpha(z)}{\beta(z)} (1-z^s)^{-d} = \sum_{k=0}^{\infty} \lambda_{d,k} z^k, \quad |z| < 1. \tag{2}$$

Definition 2.1. Let $\{Z_t\}_{t\in\mathbb{Z}}$ be a sequence of independent and identically distributed random variables (i.i.d.), with zero mean and variance equal to one, $\theta, \gamma \in \mathbb{R}$ and

$$g(Z_t) := \theta Z_t + \gamma\big[|Z_t| - \mathbb{E}\big(|Z_t|\big)\big], \quad \text{for all} \quad t \in \mathbb{Z}. \tag{3}$$

Moreover, for $\omega \in \mathbb{R}$, $d < 0.5$ and $s \in \mathbb{N}^*$, let $\{X_t\}_{t\in\mathbb{Z}}$ be the stochastic process defined by

$$X_t = \sigma_t Z_t, \tag{4}$$

$$\ln(\sigma_t^2) = \omega + \frac{\alpha(\mathcal{B})}{\beta(\mathcal{B})}(1-\mathcal{B}^s)^{-d} g(Z_{t-1}) = \omega + \lambda(\mathcal{B}) g(Z_{t-1}), \tag{5}$$

for all $t \in \mathbb{Z}$. Then, $\{X_t\}_{t\in\mathbb{Z}}$ is called a *seasonal* FIEGARCH process and it is denoted by SFIEGARCH$(p, d, q)_s$.

Example 2.1. Figure 1 shows an SFIEGARCH$(0,d,0)_s$ time series $\{X_t\}_{t=1}^{4,500}$ (left) and its conditional standard deviation $\{\sigma_t\}_{t=1}^{4,500}$ (right), obtained from expression (5), with $\omega = -2.5$, $\theta = -0.05$, $\gamma = 0.14$, $d = 0.25$, $s = 6$ and $Z_0 \sim t_{6.5}$. For this example, polynomial (2) was truncated at $m = 100,000$.

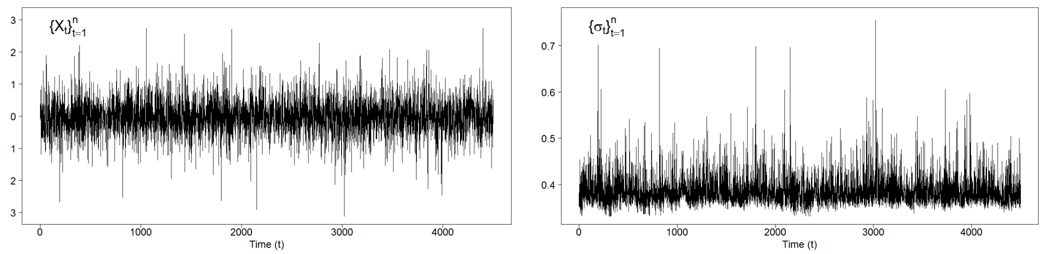

Figure 1: Simulated SFIEGARCH$(0,d,0)_s$ time series $\{X_t\}_{t=1}^{n}$ (left) and its conditional standard deviation $\{\sigma_t\}_{t=1}^{n}$ (right), with $n = 4,500$ observations.

Remark 2.1. Notice that,

- if $d = 0$, we have the so-called EGARCH(p,q) process (Nelson, 1991);
- if $s = 1$, we have the FIEGARCH(p,d,q) process (Bollerslev and Mikkelsen, 1996);
- an equivalent definition for SFIEGARCH process is given if one replaces expression (5) by

$$\beta(\mathcal{B})(1-\mathcal{B}^s)^d(\ln(\sigma_t^2)-\omega) = \alpha(\mathcal{B})g(Z_{t-1}), \quad \text{for all} \quad t \in \mathbb{Z}. \tag{6}$$

  Notice that expression (6) is similar (not necessarily equivalent) to the one presented by Bordignon et al. (2009) for a PLM-EGARCH(p,m,d,q,s) model.

The following proposition summarizes the main results regarding the stationarity and ergodicity of SFIEGARCH$(p,d,q)_s$ processes.

Proposition 2.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SFIEGARCH(p,d,q) process and suppose that γ and θ, given in (3), are not both equal to zero. Then, the following properties hold*

- $\ln(\sigma_t^2) - \omega = \sum_{k=0}^{\infty} \lambda_{d,k} g(Z_{t-1-k})$, *for all $t \in \mathbb{Z}$, is well defined and converges almost surely (a.s.) if and only if $d < 0.5$. If $d \leq 0$ the series converges absolutely a.s.*
- *If $d < 0.5$, $\{\ln(\sigma_t^2)\}_{t\in\mathbb{Z}}$ is a causal* SARFIMA$(p,0,q)\times(0,D,0)_s$ *process, with $D = d$.*
- *If $d < 0.5$, the random variable X_t is finite with probability one, for all $t \in \mathbb{Z}$.*
- *If $d < 0.5$, the stochastic processes $\{\ln(\sigma_t^2) - \omega\}_{t\in\mathbb{Z}}$, $\{\sigma_t^2\}_{t\in\mathbb{Z}}$ and $\{X_t\}_{t\in\mathbb{Z}}$ are stationary (strictly and weakly) and ergodic.*
- *If $d < 0.5$ and $\mathbb{E}(|\ln(Z_0^2)|) < \infty$, then the process $\{\ln(X_t^2)\}_{t\in\mathbb{Z}}$ is well defined and it is strictly stationary and ergodic. Moreover, if $\mathbb{E}([\ln(Z_0^2)]^2) < \infty$ then it is also weakly stationary.*

Proof. See Lopes and Prass (2012). □

Proposition 2.2 gives a recurrence formula to calculate the coefficients of the polynomial $\lambda(\cdot)$. This expression is used to generate the SFIEGARCH time series in the simulation study presented in Section 4.

Proposition 2.2. *Let $\lambda(\cdot)$ be the polynomial defined by (5). Suppose $\alpha(\cdot)$ and $\beta(\cdot)$ have no common roots and $\beta(z) \neq 0$ in the closed disk $\{z : |z| \leq 1\}$. Then, the coefficients $\lambda_{d,k}$, for all $k \in \mathbb{N}$, are given by*

$$\lambda_{d,0} = 1 \quad \text{and} \quad \lambda_{d,k} = -\alpha_k^* + \sum_{i=0}^{k-1} \lambda_{d,i} \Big[\sum_{j=0}^{\min\{k-i,q\}} \delta^*_{d,\frac{k-i-j}{s}} \beta_j \Big], \qquad \text{for all } k > 0, \tag{7}$$

where $\alpha_k^* = \alpha_k$, if $k \leq p$, and 0 otherwise, and $\delta^*_{d,m} = \delta_{d,m}$, if $m \in \mathbb{N}$, and 0 otherwise, with $\delta_{d,m}$ defined in (1), for all $m \in \mathbb{N}$.

Proof. See Lopes and Prass (2012). □

3 Quasi Maximum Likelihood Estimator (QMLE)

Let $\boldsymbol{\psi} \in \mathbb{R}^{\mathfrak{p}}$, with $\mathfrak{p} \leq \mathfrak{q}+p+q+4$, be the vector of unknown parameters in expression (5) and observe that, from (4), the standardized residual of a SFIEGARCH process is given by

$$Z_t := \frac{X_t}{\sigma_t}, \quad \text{where} \quad \sigma_t := \sigma_t(\boldsymbol{\psi}) = \exp\left\{ \frac{1}{2} \left[\omega + \sum_{k=0}^{\infty} \lambda_{d,k} g(Z_{t-1-k}) \right] \right\}, \quad \text{for all} \quad t \in \mathbb{Z}.$$

Denote by $f_Z(\cdot;\boldsymbol{\nu})$ the probability density function of Z_0, with nuisance parameters $\boldsymbol{\nu} \in \mathbb{R}^{\mathfrak{q}}$, where the value $\mathfrak{q}$ depends on the distribution considered. Thus, given an SFIEGARCH time series $\{x_t\}_{t=1}^n$, the maximum likelihood estimator (MLE) of $\boldsymbol{\psi} = (\boldsymbol{\eta}', \boldsymbol{\nu}')'$ is obtained by maximizing the log-likelihood function

$$\mathscr{L}_n(x_1, \cdots, x_n; \boldsymbol{\psi}) = \sum_{t=1}^{n} \ell_t(x_t; \boldsymbol{\psi}), \tag{8}$$

where $\ell_t(x_t;\boldsymbol{\psi}) = \ln\big(f_Z(z_t;\boldsymbol{\nu})\big) - 0.5\ln(\sigma_t^2(\boldsymbol{\psi}))$ is the log-likelihood function for the t-th observation, for all $t = 1, \cdots, n$.

Remark 3.1. Note that σ_t^2 is a $\mathcal{F}_t$-measurable function, where $\mathcal{F}_t$ is the information obtained until the instant t. Consequently, $\ell_t(\cdot;\boldsymbol{\psi})$ is the conditional distribution of $X_t|\mathcal{F}_{t-1}$, for all $t \in \mathbb{Z}$.

Obviously, the implementation of the maximum likelihood procedure requires the conditional density function $\ell_t(\cdot;\boldsymbol{\psi})$ to be known, which usually does not hold in practice. If the true distribution of Z_0 is unknown, an estimator for $\boldsymbol{\eta}$ is obtained by assuming a given distribution function for $X_t|\mathcal{F}_{t-1}$, for all $t \in \mathbb{Z}$, or equivalently, for Z_0. The estimator obtained under this condition is called pseudo-maximum likelihood estimator (PMLE) and it coincides with the MLE whenever the correct distribution function of $X_t|\mathcal{F}_{t-1}$ is specified, for all $t \in \mathbb{Z}$. Under the hypothesis that Z_0 has Gaussian distribution, it is called quasi-maximum likelihood estimator (QMLE) and the estimation procedure is called quasi-likelihood method (QLM).

Although the Student's t and the Generalized Error Distribution (GED) functions are also considered in the literature, the QLM is usually preferred. This is so because this method considers the standard Gaussian distribution and hence, the nuisance parameters do not need to be estimated. Moreover, under this assumption, $\sigma_t = \sigma_t(\boldsymbol{\eta})$, for all $t \in \mathbb{Z}$.

4 Simulation Study

We generate samples from SFIEGARCH$(0, d, 0)_s$ processes by considering two distributions for the random variable Z_0, namely, the t_{ν_1} and the GED(ν_2), with different values for the nuisance parameters ν_1 and ν_2. To estimate $\boldsymbol{\eta} = (d,\, \theta,\, \gamma)' \in \mathbb{R}^3$ we apply the quasi-maximum likelihood procedure as explained below.

4.1 Data Generating Process

To generate samples from SFIEGARCH processes we set the following:

1. the seasonal parameter $s \in \{2,\, 6\}$;
2. the differencing parameter $d \in \{0.10,\, 0.25,\, 0.35,\, 0.45\}$;
3. the nuisance parameter $\nu_1 \in \{2.5,\, 3.5,\, 5.0\}$, for the Student's t distribution, and $\nu_2 \in \{1.2,\, 2.5,\, 5.0\}$, for the GED distribution;
4. for all models, we fixed $\omega = 0$ since it is only a scaling parameter (we will assume that this parameter is known). We also considered $\theta = -0.25$ and $\gamma = 0.24$, which are values close to the ones observed in practical applications.
5. for each model, we consider the sample size $n \in \{2,000;\, 5,000\}$ with $re = 1,000$ replications;
6. when $s = 2$, the infinite sum (5) is truncated at $m = 50,000$ and, for $s = 6$, $m = 100,000$.

The sample $\{x_t\}_{t=1}^{n}$ is then obtained through the relation

$$\ln(\sigma_t^2) = \sum_{k=0}^{m} \lambda_{d,k}\, g(z_{t-1-k}) \quad \text{and} \quad x_t = \sigma_t z_t, \quad \text{for all } t = 1, \cdots, n,$$

and $\{z_t\}_{t=-m}^{n}$ is a sample of size $m + n + 1$ from the underlying distribution.

4.2 Parameter Estimation

To estimate the parameters of the model we assume that s and ω are known and hence the vector of unknown parameters is $\boldsymbol{\eta} = (d,\, \theta,\, \gamma)' \in \mathbb{R}^3$. From (8), the estimator $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ is the value that maximizes

$$\mathscr{L}(x_1, \cdots, x_n; \boldsymbol{\eta}) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{n}\left[\ln(\sigma_t^2) + \frac{x_t^2}{\sigma_t^2}\right]. \tag{9}$$

In order to solve (9), for each candidate $\boldsymbol{\eta} = (d,\, \theta,\, \gamma)' \in \mathbb{R}^3$, the coefficients $\{\lambda_{d,k}\}_{k=0}^{n}$ are obtained through the recurrence formula (7) and σ_t is obtained recursively by assuming that $g(z_t) = 0$, whenever $t < 1$ (initial conditions),

$$\sigma_1 = e^{0.5} \quad \text{and} \quad z_1 = \frac{x_1}{\sigma_1};$$
$$\sigma_t = \exp\left\{\frac{1}{2}\sum_{k=0}^{n-1} \lambda_{d,k}\, g(z_{t-1-k})\right\} \quad \text{and} \quad z_t = \frac{x_t}{\sigma_t}, \quad \text{for all } t = 2, \cdots, n.$$

4.3 Performance Measures

For any model, given $\boldsymbol{\eta} = (d, \theta, \gamma)' := (\eta_1, \eta_2, \eta_3)'$ we denote by $\hat{\eta}_i^{(k)}$ the estimate for η_i in the k-th replication, for $k \in \{1, \cdots, re\}$, $re = 1{,}000$ and $i \in \{1, 2, 3\}$. Moreover, given $\hat{\boldsymbol{\eta}}$, we calculate the mean $\bar{\eta}_i$, the standard deviation (sd), the bias ($bias$), the mean absolute error (mae) and the mean square error (mse) values,

$$\bar{\eta}_i := \frac{1}{re}\sum_{k=1}^{re} \hat{\eta}_i^{(k)}, \quad sd := \sqrt{\frac{1}{re}\sum_{k=1}^{re} (\hat{\eta}_i^{(k)} - \bar{\eta}_i)^2}, \quad bias := \frac{1}{re}\sum_{k=1}^{re} e_i^{(k)},$$

$$mae := \frac{1}{re}\sum_{k=1}^{re} |e_i^{(k)}| \quad \text{and} \quad mse := \frac{1}{re}\sum_{k=1}^{re} \left(e_i^{(k)}\right)^2,$$

where $e_i^{(k)} := \hat{\eta}_i^{(k)} - \eta_i$, for $k \in \{1, \cdots, re\}$. The values of these statistics for this simulation study are reported in Tables 1-4.

4.4 Results

Tables 1 and 2 present the estimation results for $s = 2$, when the underlying distribution functions are, respectively, the Student's t and the GED. The estimation results for $s = 6$ are shown in Tables 3 and 4.

From Tables 1-4 one observes that,

- as expected, the estimation improves (in terms of sd, mae and mse) as the sample size increases;
- generally, the parameter θ seems to be better estimated than d and γ;
- for all parameters, the $bias$ is usually negative, in the GED case, and it varies in the Student's t case;
- for the Student's t distribution, as ν_1 increases, the $bias$ and the mae (mse) decreases. This result is expected since for $\nu_1 \leq 2$, Z_0 does not have finite variance and, as $\nu_1 \to \infty$, the t_{ν_1} distribution converges to the Gaussian one;
- in the GED case, the parameters are better estimated for $\nu_2 = 2.5$ (for $\nu_2 = 2$ we have the Gaussian distribution).

Table 1: Estimation results for $s = 2$ and Student's t as the underlying distribution.

		$n = 2,000$					$n = 5,000$				
ν_1	η	$\bar{\eta}$	sd	$bias$	mae	mse	$\bar{\eta}$	sd	$bias$	mae	mse
	$d =$ 0.10	0.1868	0.2743	0.0868	0.2301	0.0828	0.1467	0.2235	0.0467	0.1698	0.0521
2.5	$\theta = -0.25$	-0.2073	0.1720	0.0427	0.1318	0.0314	-0.2223	0.1404	0.0277	0.0987	0.0205
	$\gamma =$ 0.24	0.2446	0.1801	0.0046	0.1356	0.0325	0.2274	0.1479	-0.0126	0.1035	0.0220
	$d =$ 0.10	0.0677	0.1986	-0.0323	0.1503	0.0405	0.0590	0.1221	-0.0410	0.0948	0.0166
3.5	$\theta = -0.25$	-0.2451	0.0849	0.0049	0.0633	0.0072	-0.2462	0.0573	0.0038	0.0431	0.0033
	$\gamma =$ 0.24	0.2218	0.1128	-0.0182	0.0877	0.0131	0.2240	0.0796	-0.0160	0.0602	0.0066
	$d =$ 0.10	0.0740	0.1344	-0.0260	0.1021	0.0187	0.0748	0.0784	-0.0252	0.0642	0.0068
5.0	$\theta = -0.25$	-0.2507	0.0555	-0.0007	0.0425	0.0031	-0.2501	0.0361	-0.0001	0.0281	0.0013
	$\gamma =$ 0.24	0.2279	0.0801	-0.0121	0.0639	0.0066	0.2326	0.0511	-0.0074	0.0410	0.0027
	$d =$ 0.25	0.2497	0.2355	-0.0003	0.1673	0.0555	0.2108	0.1795	-0.0392	0.1220	0.0337
2.5	$\theta = -0.25$	-0.2122	0.1650	0.0378	0.1211	0.0287	-0.2254	0.1381	0.0246	0.0917	0.0197
	$\gamma =$ 0.24	0.2288	0.1760	-0.0112	0.1346	0.0311	0.2091	0.1516	-0.0309	0.1075	0.0239
	$d =$ 0.25	0.2005	0.1592	-0.0495	0.1189	0.0278	0.1831	0.0987	-0.0669	0.0893	0.0142
3.5	$\theta = -0.25$	-0.2462	0.0825	0.0038	0.0608	0.0068	-0.2488	0.0548	0.0012	0.0408	0.0030
	$\gamma =$ 0.24	0.2122	0.1149	-0.0278	0.0908	0.0140	0.2101	0.0797	-0.0299	0.0663	0.0073
	$d =$ 0.25	0.2233	0.1031	-0.0267	0.0772	0.0113	0.2157	0.0592	-0.0343	0.0538	0.0047
5.0	$\theta = -0.25$	-0.2495	0.0540	0.0005	0.0418	0.0029	-0.2507	0.0344	-0.0007	0.0267	0.0012
	$\gamma =$ 0.24	0.2248	0.0795	-0.0152	0.0639	0.0065	0.2241	0.0498	-0.0159	0.0412	0.0027
	$d =$ 0.35	0.2899	0.2197	-0.0601	0.1576	0.0519	0.2542	0.1713	-0.0958	0.1377	0.0385
2.5	$\theta = -0.25$	-0.2238	0.1572	0.0262	0.1111	0.0254	-0.2391	0.1246	0.0109	0.0820	0.0157
	$\gamma =$ 0.24	0.2076	0.1804	-0.0324	0.1409	0.0336	0.1878	0.1448	-0.0522	0.1167	0.0237
	$d =$ 0.35	0.2874	0.1407	-0.0626	0.1147	0.0237	0.2663	0.1007	-0.0837	0.1036	0.0171
3.5	$\theta = -0.25$	-0.2522	0.0799	-0.0022	0.0584	0.0064	-0.2550	0.0603	-0.0050	0.0406	0.0037
	$\gamma =$ 0.24	0.1925	0.1214	-0.0475	0.1013	0.0170	0.1775	0.0974	-0.0625	0.0876	0.0134
	$d =$ 0.35	0.3105	0.0994	-0.0395	0.0821	0.0114	0.2974	0.0701	-0.0526	0.0723	0.0077
5.0	$\theta = -0.25$	-0.2530	0.0545	-0.0030	0.0418	0.0030	-0.2563	0.0357	-0.0063	0.0279	0.0013
	$\gamma =$ 0.24	0.2031	0.0878	-0.0369	0.0756	0.0091	0.1936	0.0629	-0.0464	0.0624	0.0061
	$d =$ 0.45	0.3492	0.2373	-0.1008	0.1828	0.0665	0.3311	0.2038	-0.1189	0.1677	0.0557
2.5	$\theta = -0.25$	-0.2428	0.1545	0.0072	0.1062	0.0239	-0.2510	0.1242	-0.0010	0.0799	0.0154
	$\gamma =$ 0.24	0.1598	0.1755	-0.0802	0.1615	0.0372	0.1267	0.1471	-0.1133	0.1578	0.0345
	$d =$ 0.45	0.3848	0.1719	-0.0652	0.1357	0.0338	0.3776	0.1411	-0.0724	0.1170	0.0252
3.5	$\theta = -0.25$	-0.2555	0.0846	-0.0055	0.0631	0.0072	-0.2569	0.0605	-0.0069	0.0457	0.0037
	$\gamma =$ 0.24	0.1240	0.1304	-0.1160	0.1475	0.0305	0.0945	0.1059	-0.1455	0.1575	0.0324
	$d =$ 0.45	0.4121	0.1360	-0.0379	0.1074	0.0199	0.4108	0.1044	-0.0392	0.0841	0.0124
5.0	$\theta = -0.25$	-0.2506	0.0600	-0.0006	0.0465	0.0036	-0.2540	0.0416	-0.0040	0.0327	0.0017
	$\gamma =$ 0.24	0.1220	0.1154	-0.1180	0.1402	0.0273	0.0937	0.0910	-0.1463	0.1516	0.0297

Table 2: Estimation results for $s = 2$ and GED as the underlying distribution.

		$n = 2,000$					$n = 5,000$				
ν_2	η	$\bar{\eta}$	sd	$bias$	mae	mse	$\bar{\eta}$	sd	$bias$	mae	mse
	$d =\ 0.10$	0.0818	0.1006	-0.0182	0.0778	0.0105	0.0828	0.0601	-0.0172	0.0482	0.0039
1.2	$\theta = -0.25$	-0.2493	0.0437	0.0007	0.0351	0.0019	-0.2481	0.0268	0.0019	0.0213	0.0007
	$\gamma =\ 0.24$	0.2355	0.0603	-0.0045	0.0480	0.0037	0.2340	0.0389	-0.0060	0.0319	0.0015
	$d =\ 0.10$	0.0942	0.0706	-0.0058	0.0556	0.0050	0.0969	0.0416	-0.0031	0.0331	0.0017
2.5	$\theta = -0.25$	-0.2515	0.0281	-0.0015	0.0226	0.0008	-0.2508	0.0177	-0.0008	0.0141	0.0003
	$\gamma =\ 0.24$	0.2406	0.0496	0.0006	0.0395	0.0025	0.2419	0.0304	0.0019	0.0242	0.0009
	$d =\ 0.10$	0.0861	0.0536	-0.0139	0.0431	0.0031	0.0865	0.0323	-0.0135	0.0276	0.0012
5.0	$\theta = -0.25$	-0.2531	0.0231	-0.0031	0.0185	0.0005	-0.2527	0.0145	-0.0027	0.0119	0.0002
	$\gamma =\ 0.24$	0.2396	0.0425	-0.0004	0.0336	0.0018	0.2388	0.0280	-0.0012	0.0224	0.0008
	$d =\ 0.25$	0.2270	0.0787	-0.0230	0.0613	0.0067	0.2217	0.0470	-0.0283	0.0433	0.0030
1.2	$\theta = -0.25$	-0.2486	0.0431	0.0014	0.0345	0.0019	-0.2491	0.0257	0.0009	0.0206	0.0007
	$\gamma =\ 0.24$	0.2299	0.0587	-0.0101	0.0475	0.0035	0.2248	0.0380	-0.0152	0.0329	0.0017
	$d =\ 0.25$	0.2465	0.0544	-0.0035	0.0405	0.0030	0.2432	0.0317	-0.0068	0.0256	0.0011
2.5	$\theta = -0.25$	-0.2525	0.0286	-0.0025	0.0228	0.0008	-0.2533	0.0175	-0.0033	0.0144	0.0003
	$\gamma =\ 0.24$	0.2407	0.0491	0.0007	0.0392	0.0024	0.2393	0.0302	-0.0007	0.0238	0.0009
	$d =\ 0.25$	0.2304	0.0430	-0.0196	0.0372	0.0022	0.2272	0.0273	-0.0228	0.0292	0.0013
5.0	$\theta = -0.25$	-0.2582	0.0227	-0.0082	0.0192	0.0006	-0.2595	0.0142	-0.0095	0.0138	0.0003
	$\gamma =\ 0.24$	0.2324	0.0425	-0.0076	0.0340	0.0019	0.2293	0.0288	-0.0107	0.0245	0.0009
	$d =\ 0.35$	0.3137	0.0868	-0.0363	0.0741	0.0088	0.3007	0.0663	-0.0493	0.0681	0.0068
1.2	$\theta = -0.25$	-0.2509	0.0446	-0.0009	0.0351	0.0020	-0.2559	0.0281	-0.0059	0.0229	0.0008
	$\gamma =\ 0.24$	0.2068	0.0731	-0.0332	0.0634	0.0064	0.1912	0.0547	-0.0488	0.0579	0.0054
	$d =\ 0.35$	0.3326	0.0740	-0.0174	0.0614	0.0058	0.3205	0.0619	-0.0295	0.0572	0.0047
2.5	$\theta = -0.25$	-0.2567	0.0313	-0.0067	0.0254	0.0010	-0.2615	0.0205	-0.0115	0.0188	0.0006
	$\gamma =\ 0.24$	0.2099	0.0725	-0.0301	0.0626	0.0062	0.1916	0.0617	-0.0484	0.0614	0.0061
	$d =\ 0.35$	0.3237	0.0651	-0.0263	0.0569	0.0049	0.3178	0.0534	-0.0322	0.0512	0.0039
5.0	$\theta = -0.25$	-0.2629	0.0248	-0.0129	0.0223	0.0008	-0.2662	0.0166	-0.0162	0.0194	0.0005
	$\gamma =\ 0.24$	0.1918	0.0707	-0.0482	0.0674	0.0073	0.1749	0.0642	-0.0651	0.0740	0.0084
	$d =\ 0.45$	0.4147	0.1352	-0.0353	0.1050	0.0195	0.4222	0.0992	-0.0278	0.0787	0.0106
1.2	$\theta = -0.25$	-0.2470	0.0509	0.0030	0.0396	0.0026	-0.2505	0.0365	-0.0005	0.0284	0.0013
	$\gamma =\ 0.24$	0.1251	0.1096	-0.1149	0.1327	0.0252	0.0875	0.0866	-0.1525	0.1558	0.0307
	$d =\ 0.45$	0.4365	0.1064	-0.0135	0.0832	0.0115	0.4367	0.0797	-0.0133	0.0646	0.0065
2.5	$\theta = -0.25$	-0.2533	0.0369	-0.0033	0.0289	0.0014	-0.2553	0.0292	-0.0053	0.0234	0.0009
	$\gamma =\ 0.24$	0.1114	0.1074	-0.1286	0.1440	0.0281	0.0778	0.0823	-0.1622	0.1657	0.0331
	$d =\ 0.45$	0.4440	0.0923	-0.0060	0.0739	0.0086	0.4419	0.0681	-0.0081	0.0543	0.0047
5.0	$\theta = -0.25$	-0.2540	0.0355	-0.0040	0.0278	0.0013	-0.2551	0.0269	-0.0051	0.0219	0.0007
	$\gamma =\ 0.24$	0.0952	0.1046	-0.1448	0.1567	0.0319	0.0688	0.0806	-0.1712	0.1732	0.0358

Table 3: Estimation results for $s = 6$ and Student's t as the underlying distribution.

		$n = 2,000$					$n = 5,000$				
ν_1	η	$\bar{\eta}$	*sd*	*bias*	*mae*	*mse*	$\bar{\eta}$	*sd*	*bias*	*mae*	*mse*
	$d =$ 0.10	0.2098	0.2774	0.1098	0.2404	0.0890	0.1726	0.2097	0.0726	0.1738	0.0493
2.5	$\theta = -0.25$	-0.2053	0.1711	0.0447	0.1329	0.0313	-0.2177	0.1248	0.0323	0.0949	0.0166
	$\gamma =$ 0.24	0.2545	0.1809	0.0145	0.1331	0.0329	0.2310	0.1361	-0.0090	0.0996	0.0186
	$d =$ 0.10	0.0690	0.2022	-0.0310	0.1510	0.0419	0.0672	0.1337	-0.0328	0.1018	0.0190
3.5	$\theta = -0.25$	-0.2462	0.0886	0.0038	0.0662	0.0079	-0.2465	0.0591	0.0035	0.0445	0.0035
	$\gamma =$ 0.24	0.2259	0.1104	-0.0141	0.0849	0.0124	0.2247	0.0728	-0.0153	0.0574	0.0055
	$d =$ 0.10	0.0767	0.1314	-0.0233	0.0990	0.0178	0.0840	0.0799	-0.0160	0.0628	0.0066
5.0	$\theta = -0.25$	-0.2456	0.0540	0.0044	0.0424	0.0029	-0.2474	0.0346	0.0026	0.0276	0.0012
	$\gamma =$ 0.24	0.2316	0.0778	-0.0084	0.0615	0.0061	0.2298	0.0501	-0.0102	0.0410	0.0026
	$d =$ 0.25	0.2800	0.2279	0.0300	0.1715	0.0528	0.2368	0.1780	-0.0132	0.1222	0.0319
2.5	$\theta = -0.25$	-0.2127	0.1663	0.0373	0.1248	0.0291	-0.2255	0.1150	0.0245	0.0863	0.0138
	$\gamma =$ 0.24	0.2390	0.1772	-0.0010	0.1299	0.0314	0.2144	0.1284	-0.0256	0.1003	0.0171
	$d =$ 0.25	0.2080	0.1650	-0.0420	0.1216	0.0290	0.2005	0.1031	-0.0495	0.0842	0.0131
3.5	$\theta = -0.25$	-0.2468	0.0848	0.0032	0.0640	0.0072	-0.2472	0.0592	0.0028	0.0433	0.0035
	$\gamma =$ 0.24	0.2157	0.1077	-0.0243	0.0857	0.0122	0.2105	0.0755	-0.0295	0.0613	0.0066
	$d =$ 0.25	0.2288	0.0990	-0.0212	0.0752	0.0102	0.2278	0.0576	-0.0222	0.0474	0.0038
5.0	$\theta = -0.25$	-0.2456	0.0524	0.0044	0.0410	0.0028	-0.2479	0.0331	0.0021	0.0263	0.0011
	$\gamma =$ 0.24	0.2267	0.0745	-0.0133	0.0602	0.0057	0.2226	0.0488	-0.0174	0.0419	0.0027
	$d =$ 0.35	0.3270	0.2099	-0.0230	0.1477	0.0446	0.2884	0.1646	-0.0616	0.1213	0.0309
2.5	$\theta = -0.25$	-0.2216	0.1604	0.0284	0.1157	0.0265	-0.2339	0.1067	0.0161	0.0783	0.0117
	$\gamma =$ 0.24	0.2221	0.1763	-0.0179	0.1319	0.0314	0.1963	0.1270	-0.0437	0.1073	0.0181
	$d =$ 0.35	0.3042	0.1447	-0.0458	0.1082	0.0230	0.2925	0.0949	-0.0575	0.0837	0.0123
3.5	$\theta = -0.25$	-0.2488	0.0810	0.0012	0.0607	0.0066	-0.2517	0.0539	-0.0017	0.0405	0.0029
	$\gamma =$ 0.24	0.1997	0.1054	-0.0403	0.0896	0.0127	0.1903	0.0734	-0.0497	0.0711	0.0079
	$d =$ 0.35	0.3247	0.0839	-0.0253	0.0664	0.0077	0.3173	0.0528	-0.0327	0.0500	0.0039
5.0	$\theta = -0.25$	-0.2479	0.0510	0.0021	0.0395	0.0026	-0.2517	0.0324	-0.0017	0.0254	0.0011
	$\gamma =$ 0.24	0.2141	0.0734	-0.0259	0.0627	0.0061	0.2064	0.0499	-0.0336	0.0486	0.0036
	$d =$ 0.45	0.3861	0.2033	-0.0639	0.1512	0.0454	0.3596	0.1705	-0.0904	0.1405	0.0372
2.5	$\theta = -0.25$	-0.2376	0.1594	0.0124	0.1079	0.0256	-0.2484	0.1034	0.0016	0.0719	0.0107
	$\gamma =$ 0.24	0.1775	0.1739	-0.0625	0.1425	0.0342	0.1512	0.1275	-0.0888	0.1306	0.0241
	$d =$ 0.45	0.3926	0.1492	-0.0574	0.1175	0.0256	0.3926	0.1112	-0.0574	0.0945	0.0157
3.5	$\theta = -0.25$	-0.2524	0.0789	-0.0024	0.0585	0.0062	-0.2546	0.0565	-0.0046	0.0428	0.0032
	$\gamma =$ 0.24	0.1540	0.1149	-0.0860	0.1186	0.0206	0.1303	0.0943	-0.1097	0.1224	0.0209
	$d =$ 0.45	0.4147	0.1062	-0.0353	0.0867	0.0125	0.4094	0.0821	-0.0406	0.0730	0.0084
5.0	$\theta = -0.25$	-0.2521	0.0533	-0.0021	0.0414	0.0028	-0.2555	0.0372	-0.0055	0.0297	0.0014
	$\gamma =$ 0.24	0.1602	0.0993	-0.0798	0.1046	0.0162	0.1350	0.0867	-0.1050	0.1136	0.0185

Table 4: Estimation results for $s = 6$ and GED as the underlying distribution.

		$n = 2,000$					$n = 5,000$				
ν_2	η	$\bar{\eta}$	sd	$bias$	mae	mse	$\bar{\eta}$	sd	$bias$	mae	mse
	$d =$ 0.10	0.0887	0.1065	-0.0113	0.0826	0.0115	0.0878	0.0602	-0.0122	0.0486	0.0038
1.2	$\theta = -0.25$	-0.2475	0.0425	0.0025	0.0342	0.0018	-0.2477	0.0271	0.0023	0.0220	0.0007
	$\gamma =$ 0.24	0.2321	0.0617	-0.0079	0.0500	0.0039	0.2340	0.0373	-0.0060	0.0304	0.0014
	$d =$ 0.10	0.0993	0.0670	-0.0007	0.0517	0.0045	0.0983	0.0421	-0.0017	0.0328	0.0018
2.5	$\theta = -0.25$	-0.2510	0.0288	-0.0010	0.0230	0.0008	-0.2509	0.0184	-0.0009	0.0147	0.0003
	$\gamma =$ 0.24	0.2376	0.0485	-0.0024	0.0389	0.0024	0.2393	0.0310	-0.0007	0.0248	0.0010
	$d =$ 0.10	0.0875	0.0523	-0.0125	0.0430	0.0029	0.0873	0.0329	-0.0127	0.0278	0.0012
5.0	$\theta = -0.25$	-0.2523	0.0235	-0.0023	0.0185	0.0006	-0.2525	0.0144	-0.0025	0.0115	0.0002
	$\gamma =$ 0.24	0.2374	0.0450	-0.0026	0.0360	0.0020	0.2378	0.0279	-0.0022	0.0227	0.0008
	$d =$ 0.25	0.2350	0.0785	-0.0150	0.0614	0.0064	0.2306	0.0438	-0.0194	0.0375	0.0023
1.2	$\theta = -0.25$	-0.2474	0.0410	0.0026	0.0330	0.0017	-0.2480	0.0263	0.0020	0.0213	0.0007
	$\gamma =$ 0.24	0.2274	0.0592	-0.0126	0.0488	0.0037	0.2270	0.0360	-0.0130	0.0310	0.0015
	$d =$ 0.25	0.2484	0.0497	-0.0016	0.0390	0.0025	0.2461	0.0304	-0.0039	0.0241	0.0009
2.5	$\theta = -0.25$	-0.2522	0.0283	-0.0022	0.0228	0.0008	-0.2524	0.0180	-0.0024	0.0145	0.0003
	$\gamma =$ 0.24	0.2370	0.0466	-0.0030	0.0371	0.0022	0.2382	0.0298	-0.0018	0.0239	0.0009
	$d =$ 0.25	0.2353	0.0417	-0.0147	0.0349	0.0020	0.2320	0.0257	-0.0180	0.0250	0.0010
5.0	$\theta = -0.25$	-0.2557	0.0229	-0.0057	0.0186	0.0006	-0.2573	0.0140	-0.0073	0.0127	0.0003
	$\gamma =$ 0.24	0.2330	0.0440	-0.0070	0.0356	0.0020	0.2317	0.0277	-0.0083	0.0234	0.0008
	$d =$ 0.35	0.3274	0.0705	-0.0226	0.0573	0.0055	0.3188	0.0444	-0.0312	0.0439	0.0029
1.2	$\theta = -0.25$	-0.2496	0.0399	0.0004	0.0318	0.0016	-0.2525	0.0263	-0.0025	0.0211	0.0007
	$\gamma =$ 0.24	0.2154	0.0595	-0.0246	0.0515	0.0041	0.2089	0.0404	-0.0311	0.0409	0.0026
	$d =$ 0.35	0.3389	0.0522	-0.0111	0.0425	0.0028	0.3308	0.0401	-0.0192	0.0360	0.0020
2.5	$\theta = -0.25$	-0.2559	0.0289	-0.0059	0.0238	0.0009	-0.2584	0.0185	-0.0084	0.0163	0.0004
	$\gamma =$ 0.24	0.2215	0.0521	-0.0185	0.0441	0.0031	0.2149	0.0431	-0.0251	0.0386	0.0025
	$d =$ 0.35	0.3300	0.0487	-0.0200	0.0416	0.0028	0.3239	0.0364	-0.0261	0.0359	0.0020
5.0	$\theta = -0.25$	-0.2600	0.0229	-0.0100	0.0199	0.0006	-0.2641	0.0144	-0.0141	0.0167	0.0004
	$\gamma =$ 0.24	0.2133	0.0524	-0.0267	0.0461	0.0035	0.2034	0.0436	-0.0366	0.0449	0.0032
	$d =$ 0.45	0.4150	0.0961	-0.0350	0.0796	0.0105	0.4130	0.0765	-0.0370	0.0686	0.0072
1.2	$\theta = -0.25$	-0.2519	0.0449	-0.0019	0.0357	0.0020	-0.2553	0.0337	-0.0053	0.0268	0.0012
	$\gamma =$ 0.24	0.1607	0.0949	-0.0793	0.1011	0.0153	0.1345	0.0859	-0.1055	0.1140	0.0185
	$d =$ 0.45	0.4289	0.0789	-0.0211	0.0647	0.0067	0.4270	0.0639	-0.0230	0.0540	0.0046
2.5	$\theta = -0.25$	-0.2577	0.0334	-0.0077	0.0268	0.0012	-0.2593	0.0243	-0.0093	0.0209	0.0007
	$\gamma =$ 0.24	0.1565	0.0995	-0.0835	0.1081	0.0169	0.1295	0.0929	-0.1105	0.1251	0.0209
	$d =$ 0.45	0.4339	0.0770	-0.0161	0.0621	0.0062	0.4303	0.0620	-0.0197	0.0519	0.0042
5.0	$\theta = -0.25$	-0.2574	0.0287	-0.0074	0.0237	0.0009	-0.2608	0.0218	-0.0108	0.0196	0.0006
	$\gamma =$ 0.24	0.1457	0.1059	-0.0943	0.1195	0.0201	0.1203	0.0964	-0.1197	0.1340	0.0236

5 Conclusions

In this work we considered seasonal FIEGARCH (SFIEGARCH) processes which account for both, the long memory and seasonal behavior observed in financial time series. Properties of these processes where discussed and a Monte Carlo simulation study was conducted to access the finite sample performance of the quasi-maximum likelihood (QML) procedure.

The samples from the SFIEGARCH model were obtained by considering a recurrence formula to calculate the coefficients $\{\lambda_{d,k}\}_{k\in\mathbb{N}}$. Two distributions were considered for the random variable Z_0, namely, the t_{ν_1} and the GED(ν_2), with different values for the nuisance parameters ν_1 and ν_2, respectively. The values of ν_1 and ν_2 were selected so we obtained lighter and heavier tails than the Gaussian distribution.

We conclude that, given the complexity of SFIEGARCH models and the misspecification of the underlying distribution, the quasi-likelihood method performs relatively well, which is indicated by the small *bias*, *mae* and *mse* values for the estimates even when $s = 6$ and the sample size is $n = 2,000$.

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