Properties of Seasonal Long Memory Processes

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Abstract

We consider the fractional ARIMA process with seasonality s, denoted by SARFIMA $(p, d, q) \times (P, D, Q)_s$, which describes time series with long memory periodical behavior at finite number of spectrum frequencies. We present the proof of several properties of these processes, such as the spectral density function expression and its behavior near the seasonal frequencies, the stationarity, the intermediate and long memory characteristics, the autocovariance function and its asymptotic expression. We also investigate the ergodicity and we present necessary and sufficient conditions for the causality and the invertibility properties of SARFIMA processes.

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1 Introduction

Recently, the study of time series turned the attention to the ones having long memory characteristics. The ARFIMA(p, d, q) process, first introduced by Granger and Joyeux (1980), and Hosking (1981 and 1984), present this property when the differencing parameter d is in the interval $(0, 0.5)$. This feature is reflected by the hyperbolic decay of its autocorrelation function or by the unboundedness of its spectral density function, while in the ARMA model, dependency between observations decays at a geometric rate.

We consider processes with long memory and periodicity characteristics, the so-called SARFIMA$(p, d, q) \times (P, D, Q)_s$ processes.

The papers by Porter-Hudak (1990), Ray (1993), Ooms (1995), Carlin and Dempster (1989), and Montanari et al. (2000) deal with seasonality analysis for observable data in different fields of applications. The work by Hassler (1994) presents a complete generalization of fractional differencing processes with the presence of periodicity considering rigid, and flexible models. It also illustrates the risk of fractional misspecification. The paper by Peiris and Singh (1996) deals with prediction, and minimum mean squared error predictors of one step ahead for seasonal fractionally integrated models. The paper by Reisen and Lopes (1999) presents forecasting results for the ARFIMA$(2, d, 2)$ model, including the variance of the mean squared error using the smoothed periodogram regression method for estimating the parameter d. The later paper also presents an analysis for a real observed data comparing the performance of both ARIMA, and ARFIMA models. Ray (1993) forecasts the IBM product revenues using a complete SARFIMA$(p, d, q) \times (P, D, Q)_s$ process. Palma (2007) uses the Kalman filter approach in

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the estimation of the parameters d and D for SARFIMA$(0,d,0)\times(0,D,0)_s$ process. A companion paper, Bisognin and Lopes (2008), to be published elsewhere, gives an extensive Monte Carlo simulation study for different estimation methods for these processes. This paper also deals with forecasting on seasonal long memory processes and an application to the theory.

The main goal of this paper is to present several theoretical properties of the SARFIMA$(p,d,q)\times(P,D,Q)_s$ processes, such as the expression of the spectral density function and its behavior near the seasonal frequencies, the stationarity, the intermediate and long memory characteristics, the autocovariance function and its asymptotic expression. We also investigate the ergodicity and we present necessary and sufficient conditions for the causality and the invertibility of SARFIMA processes.

The paper is organized as follows: the next section gives some definitions, and the proof of several properties for the SARFIMA$(p,d,q)\times(P,D,Q)_s$ processes. In Section 3, we investigate necessary and sufficient conditions for the causality and the invertibility of SARFIMA processes. The mean square and probability one convergences are given for both infinite moving average and infinite autoregressive representations. Section 4 gives the ergodicity property for these processes and Section 5 presents our final conclusions.

2 SARFIMA$(p,d,q)\times(P,D,Q)_s$ Processes

In many practical situations time series exhibit a periodic pattern. These time series are very common in meteorology, economics, hydrology, and astronomy. Sometimes, even in these fields, the period of the seasonality can depend on time, that is, the autocorrelation structure of the data varies from season to season. Here, in our analysis, we consider the seasonality period constant over seasons.

We shall consider the *seasonal autoregressive fractionally integrated moving average* process, denoted hereafter by SARFIMA$(p,d,q)\times(P,D,Q)_s$, which is an extension of the ARFIMA(p,d,q) process, proposed by Granger and Joyeux (1980) and Hosking (1981).

The following sub-section presents some definitions and properties of these processes.

2.1 Some Definitions and Properties

Definition 2.1. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary stochastic process with spectral density function $f_X(\cdot)$. Suppose there exists a real number $b\in(0,1)$, a constant $C_f>0$ and one frequency $G\in[0,\pi]$ (or a finite number of frequencies) such that

$$f_X(w)\sim C_f|w-G|^{-b},\ \text{when}\ w\to G.$$

Then, $\{X_t\}_{t\in\mathbb{Z}}$ is a *long memory process.*

Remark 2.1. In Definition 2.1, when $b\in(-1,0)$, we say that the process $\{X_t\}_{t\in\mathbb{Z}}$ has the *intermediate dependence property. We refer Doukhan et al. (2003) for more details.*

Definition 2.2. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stochastic process given by the expression

$$\phi(\mathcal{B})\Phi(\mathcal{B}^s)\nabla^d\nabla_s^D(X_t-\mu)=\theta(\mathcal{B})\Theta(\mathcal{B}^s)\varepsilon_t,\ \text{for}\ t\in\mathbb{Z},\tag{2.1}$$

where μ is the mean of the process, $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma_\varepsilon^2:=\mathbb{E}(\varepsilon_t^2)$, $s\in\mathbb{N}$ is the seasonal period, $\mathcal{B}$ is the backward-shift operator, that is, $\mathcal{B}^{sk}(X_t)=$

X_{t-sk}, $\nabla_s^D := (1-\mathcal{B}^s)^D$ is the seasonal difference operator, $\phi(\cdot)$, $\theta(\cdot)$, $\Phi(\cdot)$, and $\Theta(\cdot)$ are the polynomials of degrees p, q, P, and Q, respectively, defined by

$$\begin{aligned}\phi(\mathcal{B}) &= \sum_{i=0}^{p}(-\phi_i)\,\mathcal{B}^i, \quad \theta(\mathcal{B}) = \sum_{j=0}^{q}(-\theta_j)\,\mathcal{B}^j, \\ \Phi(\mathcal{B}) &= \sum_{k=0}^{P}(-\Phi_k)\,\mathcal{B}^k, \quad \Theta(\mathcal{B}) = \sum_{l=0}^{Q}(-\Theta_l)\,\mathcal{B}^l, \end{aligned} \tag{2.2}$$

where ϕ_i, $1 \leq i \leq p$, θ_j, $1 \leq j \leq q$, Φ_k, $1 \leq k \leq P$, and Θ_l, $1 \leq l \leq Q$ are constants and $\phi_0 = \Phi_0 = -1 = \theta_0 = \Theta_0$. Then, $\{X_t\}_{t\in\mathbb{Z}}$ is a *seasonal fractionally integrated* ARMA process with period s, denoted by SARFIMA$(p,d,q)\times(P,D,Q)_s$, where d and D are, respectively, the *differencing* and the *seasonal differencing parameters.*

Remark 2.2. *(1) The seasonal difference operator $\nabla_s^D \equiv (1-\mathcal{B}^s)^D$, with seasonality $s \in \mathbb{N}$, for all $D > -1$, is defined by means of the binomial expansion $\nabla_s^D := \sum_{k\geq 0}\binom{D}{k}(-\mathcal{B}^s)^k$, where*

$$\binom{D}{k} = \Gamma(1+D)/[\Gamma(1+k)\Gamma(1+D-k)].$$

The difference operator *∇^d is obtained when $D = d$ and $s = 1$.*

(2) A particular case of the process given by (2.1) *is when $P = p = 0 = q = Q$ and $d = 0$. This process is called the pure seasonal fractionally integrated model with period s, denoted by SARFIMA$(0,D,0)_s$ and it is given by*

$$\nabla_s^D(X_t-\mu) \equiv (1-\mathcal{B}^s)^D(X_t-\mu) = \varepsilon_t,\ t \in \mathbb{Z}. \tag{2.3}$$

We refer the reader to Bisognin and Lopes (2007) for the estimation and forecasting analysis of these processes. We also refer the reader to Brietzke et al. (2005) for a closed formula for the Durbin-Levinson's algorithm relating the partial autocorrelation and the autocorrelation functions of these processes.

(3) When $P = 0 = Q$, $D = 0$ and $s = 1$ the SARFIMA$(p,d,q)\times(P,D,Q)_s$ process is just the ARFIMA(p,d,q) process (see Beran, 1994). In this case we already know the behavior of the parameter estimators (see Reisen and Lopes, 1999 and Lopes, 2008).

(4) When $p = 0 = q$, $D = 0 = d$ the SARFIMA$(p,d,q)\times(P,D,Q)_s$ process is reduced to the SARMA$(p,q)_s$ process.

(5) The negative value for the coefficients of all four polynomials in expression (2.2) *is used in the companion paper Bisognin and Lopes (2008). For this reason we maintain it here.*

It is convenient to introduce the notation $\mathbb{Z}_{\geqslant} = \{k \in \mathbb{Z} | k \geqslant 0\}$, $\mathbb{Z}_{\leqslant} = \{k \in \mathbb{Z} | k \leqslant 0\}$ and $A = \{1,\cdots,s-1\} \subset \mathbb{N}$.

In the following theorem we present the expression of the spectral density function and its behavior near the seasonal frequencies, the stationarity, the intermediate and long memory properties and the autocovariance function for the SARFIMA$(0,d,0)\times(0,D,0)_s$ processes.

Theorem 2.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be the SARFIMA$(0,d,0)\times(0,D,0)_s$ process given by the expression (2.1), with zero mean, $s \in \mathbb{N}$ as the seasonal period and $P = p = 0 = q = Q$. Then, the following is true.*

(i) *When* $|d+D|<0.5$ *and* $|D|<0.5$*, the process* $\{X_t\}_{t\in\mathbb{Z}}$ *has spectral density function given by*

$$f_X(w) = \frac{\sigma_\varepsilon^2}{2\pi}\left[2\,\sin\left(\frac{w}{2}\right)\right]^{-2d}\left[2\,\sin\left(\frac{sw}{2}\right)\right]^{-2D},\ 0<w\leqslant\pi. \tag{2.4}$$

Its behavior near the seasonal frequencies is given by the expressions (2.8) *and* (2.10)*.*

(ii) *When* $d+D<0.5$ *and* $D<0.5$*,* $\{X_t\}_{t\in\mathbb{Z}}$ *is a stationary process.*

(iii) *When* $0<d+D<0.5$ *and* $0<D<0.5$*, the process* $\{X_t\}_{t\in\mathbb{Z}}$ *has long memory property.*

(iv) *When* $-0.5<d+D<0$ *and* $-0.5<D<0$*, the process* $\{X_t\}_{t\in\mathbb{Z}}$ *has intermediate memory property.*

(v) *The process* $\{X_t\}_{t\in\mathbb{Z}}$ *has autocovariance function of order* h*,* $h\in\mathbb{Z}_{\geqslant}$*, given by*

$$\gamma_X(h) = \begin{cases} \sigma_\varepsilon^2 \displaystyle\sum_{\nu\in\mathbb{Z}_{\geqslant}} \gamma_Z(s\nu)\gamma_Y(h-s\nu), & \text{if } h=s\ell,\ \ \ell\in\mathbb{Z}_{\geqslant};\\[2ex] 0, & \text{if } h=s\ell+\zeta,\ \ \zeta\in A, \end{cases} \tag{2.5}$$

where $A=\{1,\cdots,s-1\}$*. The process* $\{Z_t\}_{t\in\mathbb{Z}}$ *is a* $\text{SARFIMA}(0,D,0)_s$ *(see equation (2.3)) with autocovariance function of order* υ*,* $\upsilon\in\mathbb{Z}_{\geqslant}$*, given by*

$$\gamma_Z(s\upsilon+\xi) = \begin{cases} \dfrac{(-1)^\upsilon\Gamma(1-2D)}{\Gamma(\upsilon-D+1)\Gamma(1-\upsilon-D)} = \gamma_X(\upsilon), & \text{if } \xi=0,\\[2ex] 0, & \text{if } \xi\in A, \end{cases} \tag{2.6}$$

and the process $\{Y_t\}_{t\in\mathbb{Z}}$ *is an* $\text{ARFIMA}(0,d,0)$ *(see item (3) in Remark 2.2, when* $p=0=q$*) with autocovariance function of order* h*,* $h\in\mathbb{Z}_{\geqslant}$*, given by*

$$\gamma_Y(h) = \frac{(-1)^h\Gamma(1-2d)}{\Gamma(h-d+1)\Gamma(1-h-d)}. \tag{2.7}$$

Proof. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a $\text{SARFIMA}(0,d,0)\times(0,D,0)_s$ process, given by the expression (2.1) when $P=p=0=q=Q$, with seasonality $s\in\mathbb{N}$.

(i) From the spectral density function definition, for any stationary process, one has

$$f_X(w) = \frac{\sigma_\varepsilon^2}{2\pi}\left|1-e^{-iw}\right|^{-2d}\left|1-e^{-isw}\right|^{-2D},\ \text{for } 0<w\leqslant\pi.$$

Since $\lim\limits_{w\to 0}\dfrac{\sin(sw)}{sw}=1$, one has $\sin(sw)\sim sw$, when $w\to 0$. Then,

$$f_X(w)\sim C_1|w-w_0|^{-2(d+D)}, \quad \text{when} \quad w\to 0, \tag{2.8}$$

where $w_0=0$ and $C_1=\frac{\sigma_\varepsilon^2}{2\pi}s^{-2D}$.

For all $j = 1, \cdots, \lfloor s/2 \rfloor$, where $\lfloor x \rfloor$ means the integer part of x, when $\lambda \to 0$, one has

$$\begin{aligned} f_X(\lambda + w_j) &= \frac{\sigma_\varepsilon^2}{2\pi} \left| 2\sin\left(\frac{\lambda}{2} + \frac{w_j}{2}\right) \right|^{-2d} \left| 2\sin\left(\frac{s\lambda}{2} + \frac{sw_j}{2}\right) \right|^{-2D} \\ &\sim \frac{\sigma_\varepsilon^2}{2\pi} \left| 2\sin\left(\frac{w_j}{2}\right) \right|^{-2d} s^{-2D} |\lambda|^{-2D} \\ &= C_2 |\lambda|^{-2D}, \end{aligned} \tag{2.9}$$

where $w_j = \frac{2\pi j}{s}$, for all $j = 1, \cdots, \lfloor s/2 \rfloor$, and $C_2 = \frac{\sigma_\varepsilon^2}{2\pi} s^{-2D} \left| 2 \ \sin\left(\frac{w_j}{2}\right) \right|^{-2d}$.

In expression (2.9), if $\lambda = w - w_j$, for $1 \leqslant j \leqslant \lfloor s/2 \rfloor$, one has

$$f_X(w) \sim C_2 |w - w_j|^{-2D}, \quad \text{when} \quad w \to w_j. \tag{2.10}$$

(ii) Let $f_X(\cdot)$, given by the expression (2.4), be the spectral density function of the $\{X_t\}_{t \in \mathbb{Z}}$ process. From item (i) of this theorem, $f_X(w) = f_X(-w)$ and $f_X(w) \geqslant 0$. Therefore, the process $\{X_t\}_{t \in \mathbb{Z}}$ is stationary if

$$\int_{-\pi}^{\pi} f_X(w) dw = 2 \int_0^{\pi} f_X(w) dw < \infty. \tag{2.11}$$

The singularity values for the spectral density function occur in the seasonal frequencies $w_j = \frac{2\pi j}{s}$, for all $j = 0, 1, \cdots, \lfloor s/2 \rfloor$.

From item (i), one has

$$C_1 \int_0^{\pi} |w|^{-2(d+D)} dw < \infty \, \text{and} \, C_2 \int_0^{\pi} |w - w_j|^{-2D} dw < \infty,$$

when $d + D < 0.5$ and $D < 0.5$, respectively. Thus, when $d + D < 0.5$ and $D < 0.5$, $\int_{-\pi}^{\pi} f_X(w) dw = 2 \int_0^{\pi} f_X(w) dw < \infty$. Therefore, from Herglotz' theorem (see Brockwell and Davis, 1991), the process $\{X_t\}_{t \in \mathbb{Z}}$ is stationary when $d + D < 0.5$ and $D < 0.5$.

(iii) From the asymptotic expression of the spectral density function of a SARFIMA$(0, d, 0) \times (0, D, 0)_s$ process, from item (i) of this theorem, and Definition 2.1, the process $\{X_t\}_{t \in \mathbb{Z}}$ has long memory property when $0 < d + D < 0.5$ and $0 < D < 0.5$.

(iv) From Remark 2.1, the process $\{X_t\}_{t \in \mathbb{Z}}$ has intermediate memory property when $-0.5 < d + D < 0$ and $-0.5 < D < 0$.

(v) Let $\{X_t\}_{t \in \mathbb{Z}}$ be a causal and stationary SARFIMA$(0, d, 0) \times (0, D, 0)_s$ process, given by the expression (2.1) when $P = p = 0 = q = Q$, with seasonality $s \in \mathbb{N}$. One wants to find the autocovariance function of the process $\{X_t\}_{t \in \mathbb{Z}}$. Let $\{Z_t\}_{t \in \mathbb{Z}}$ be a SARFIMA$(0, D, 0)_s$ process (see item (2) in Remark 2.2), given by

$$(1 - \mathcal{B}^s)^D Z_t = \epsilon_t^*, \quad \text{for} \quad t \in \mathbb{Z},$$

where $\{\epsilon_t^*\}_{t \in \mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma_{\epsilon^*}^2 = \mathbb{E}((\epsilon_t^*)^2) < \infty$.

The process $\{Z_t\}_{t\in\mathbb{Z}}$ has an infinite moving average representation given by

$$Z_t = (1-\mathcal{B}^s)^{-D}\epsilon_t^* = \sum_{j\in\mathbb{Z}_\geqslant} \psi_j \mathcal{B}^{sj}(\epsilon_t^*) = \sum_{j\in\mathbb{Z}_\geqslant} \psi_j \epsilon_{t-sj}^*, \tag{2.12}$$

where ψ_j is given by the expression (2.17) below, for all $j \in \mathbb{Z}_\geqslant$. From Proposition 3.1 (see 'Causality and Invertibility Properties' Section) and Proposition 3.1.2 in Brockwell and Davis (1991), one has

$$\gamma_Z(h) = \mathrm{Cov}(Z_{t+h}, Z_t) = \sum_{j\in\mathbb{Z}_\geqslant}\sum_{\tau\in\mathbb{Z}_\geqslant} \psi_j\psi_\tau\gamma_{\epsilon^*}(h - sj + s\tau), \tag{2.13}$$

where $\gamma_{\epsilon^*}(\cdot)$ is the autocovariance function of the process $\{\epsilon_t^*\}_{t\in\mathbb{Z}}$.
When $h - sj + s\tau = 0$, one has $j = \frac{h}{s} + \tau$. Therefore, equation (2.13) can be rewritten as

$$\gamma_Z(h) \;=\; \sigma_{\epsilon^*}^2 \sum_{\tau\in\mathbb{Z}_\geqslant} \psi_{\frac{h}{s}+\tau}\psi_\tau. \tag{2.14}$$

Taking $h = s\ell$, for $\ell \in \mathbb{Z}_\geqslant$, one has

$$\gamma_Z(s\ell) = \sigma_{\epsilon^*}^2 \sum_{\tau\in\mathbb{Z}_\geqslant} \psi_{\ell+\tau}\psi_\tau.$$

In equation (2.14), if $h = s\ell + \zeta$, $\zeta \in A$, where $A = \{1, \cdots, s-1\}$, $\gamma_Z(h) = 0$. Therefore, the autocovariance function of the process $\{Z_t\}_{t\in\mathbb{Z}}$ is given by

$$\gamma_Z(h) = \begin{cases} \sigma_{\epsilon^*}^2 \displaystyle\sum_{\tau\in\mathbb{Z}_\geqslant} \psi_{\ell+\tau}\psi_\tau, & \text{if } \ h = s\ell, \ \ \ell \in \mathbb{Z}_\geqslant; \\ 0, & \text{if } \ h = s\ell + \zeta, \ \ \zeta \in A. \end{cases} \tag{2.15}$$

For $d, D \in (-0.5, 0.5)$, let $\{X_t\}_{t\in\mathbb{Z}}$ be given as $(1-\mathcal{B}^s)^{-D}Y_t$, for all $t \in \mathbb{Z}$, with innovation process $\{Y_t\}_{t\in\mathbb{Z}}$, given by $Y_t = (1-\mathcal{B})^{-d}\epsilon_t'$ (that is, $\{Y_t\}_{t\in\mathbb{Z}}$ is an ARFIMA$(0, d, 0)$ process with innovation process $\{\epsilon_t'\}_{t\in\mathbb{Z}}$, which is a white noise process with zero mean and variance $\sigma_{\epsilon'}^2$). Thus,

$$X_t \;=\; (1-\mathcal{B}^s)^{-D}Y_t = \sum_{j\in\mathbb{Z}_\geqslant} \psi_j Y_{t-sj}, \tag{2.16}$$

where

$$\psi_j := \begin{cases} \dfrac{\Gamma(j+D)}{\Gamma(j+1)\Gamma(D)}, & \text{for } \ j \in \mathbb{Z}_\geqslant, \\ 0, & \text{for } \ j \notin \mathbb{Z}_\geqslant. \end{cases} \tag{2.17}$$

For $D < 0.5$, one has $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j| < \infty$ and $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j|^2 < \infty$ (see Lemma 3.1 in Section 3). When $d < 0.5$, the process $\{Y_t\}_{t\in\mathbb{Z}}$ is stationary, that is, $\sup_t \mathbb{E}|Y_t|^2 < \infty$. Therefore, the series in expression (2.16) converges in mean square sense (see Brietzke et al., 2005).

Then, the autocovariance function of the process $\{X_t\}_{t\in\mathbb{Z}}$ is given by

$$\begin{aligned}\gamma_X(h) &= \mathrm{Cov}(X_{t+h}, X_t) = \mathrm{Cov}\left(\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j Y_{t+h-sj}, \sum_{\tau\in\mathbb{Z}_{\geqslant}} \psi_\tau Y_{t-s\tau}\right) \\ &= \sum_{j\in\mathbb{Z}_{\geqslant}} \sum_{\tau\in\mathbb{Z}_{\geqslant}} \psi_j\psi_\tau \mathrm{Cov}(Y_{t+h-sj}, Y_{t-s\tau}) \\ &= \sigma^2_{\epsilon'} \sum_{j\in\mathbb{Z}_{\geqslant}} \sum_{\tau\in\mathbb{Z}_{\geqslant}} \psi_j\psi_\tau\gamma_Y(h - s(j-\tau)), \end{aligned} \tag{2.18}$$

where $\gamma_Y(\cdot)$ is the autocovariance function of the process $\{Y_t\}_{t\in\mathbb{Z}}$ (see equation (2.7)). Taking $\nu = j - \tau$, in equation (2.18), we get

$$\gamma_X(h) = \sigma^2_{\epsilon'} \sum_{\nu\geqslant -\tau} \sum_{\tau\in\mathbb{Z}_{\geqslant}} \psi_{\nu+\tau}\psi_\tau\gamma_Y(h - s\nu). \tag{2.19}$$

By definition of the ψ_j's coefficients (see expression (2.17)), the equation (2.19) becomes

$$\gamma_X(h) = \sigma^2_{\epsilon'} \sum_{\nu\in\mathbb{Z}_{\geqslant}} \sum_{\tau\in\mathbb{Z}_{\geqslant}} \psi_{\nu+\tau}\psi_\tau\gamma_Y(h - s\nu). \tag{2.20}$$

Applying equation (2.15) into (2.20), one has the expression (2.5), which is the autocovariance function of the SARFIMA$(0, d, 0) \times (0, D, 0)_s$ process, where $\sigma^2_\varepsilon = \sigma^2_{\epsilon'}/\sigma^2_{\epsilon^*}$ is the variance of the white noise process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$, $\{Z_t\}_{t\in\mathbb{Z}}$ is a SARFIMA$(0, D, 0)_s$ process and $\{Y_t\}_{t\in\mathbb{Z}}$ is an ARFIMA$(0, d, 0)$ process, with $\gamma_Z(\cdot)$ and $\gamma_Y(\cdot)$ given, respectively, by expressions (2.6) and (2.7).

□

Theorem 2.2 below presents some properties of SARFIMA$(p, d, q) \times (P, D, Q)_s$ processes.

Theorem 2.2. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a* SARFIMA$(p, d, q) \times (P, D, Q)_s$ *process given by the expression (2.1), with zero mean and seasonal period $s \in \mathbb{N}$. Suppose $\phi(z)\Phi(z^s) = 0$ and $\theta(z)\Theta(z^s) = 0$ have no common zeroes. Then, the following is true.*

(i) *If $|d + D| < 0.5$ and $|D| < 0.5$, the process $\{X_t\}_{t\in\mathbb{Z}}$ has spectral density function given by*

$$f_X(w) = \frac{\sigma^2_\varepsilon}{2\pi} \frac{|\theta(e^{-iw})|^2|\Theta(e^{-isw})|^2}{|\phi(e^{-iw})|^2|\Phi(e^{-isw})|^2} \left|2\sin\left(\frac{w}{2}\right)\right|^{-2d} \left|2\sin\left(\frac{sw}{2}\right)\right|^{-2D}, \tag{2.21}$$

for $0 < w \leqslant \pi$.

Its behavior near to seasonal frequencies is given by expressions (2.25) *and* (2.29).

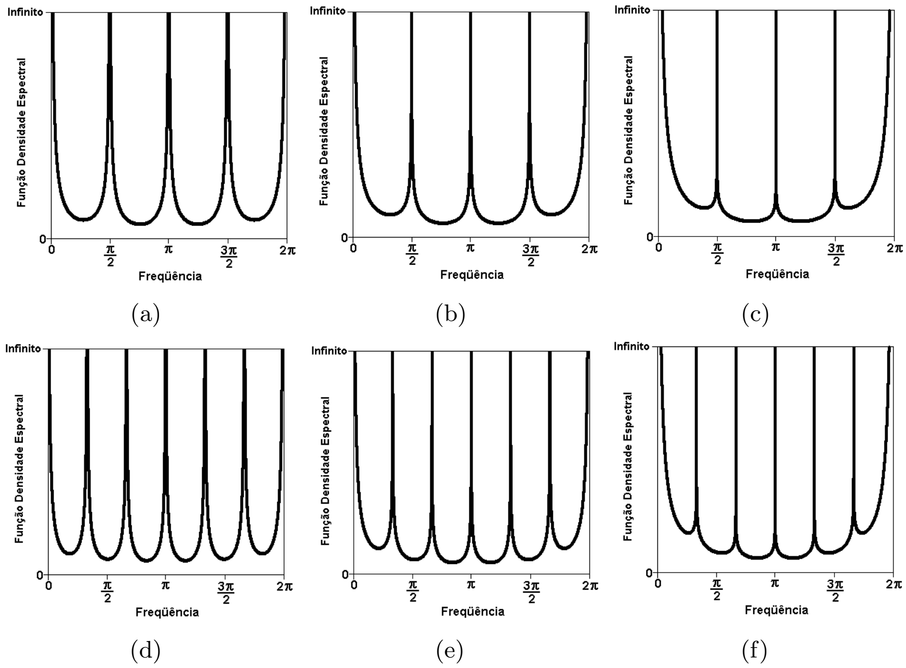

Figure 2.1: Spectral Density Function of a SARFIMA$(0,d,0)\times(0,D,0)_s$ Process: (a) $d=0.1$, $D=0.35$ and $s=4$; (b) $d=0.2$, $D=0.2$ and $s=4$; (c) $d=0.35$, $D=0.1$ and $s=4$; (d) $d=0.1$, $D=0.35$ and $s=6$; (e) $d=0.2$, $D=0.2$ and $s=6$; (f) $d=0.35$, $D=0.1$ and $s=6$.

(ii) *The process* $\{X_t\}_{t\in\mathbb{Z}}$ *is stationary if* $d+D<0.5$, $D<0.5$ *and* $\phi(z)\Phi(z^s)\neq 0$, *for* $|z|\leqslant 1$.

(iii) *The stationary process* $\{X_t\}_{t\in\mathbb{Z}}$ *has long memory property if* $0<d+D<0.5$, $0<D<0.5$ *and* $\phi(z)\Phi(z^s)\neq 0$, *for* $|z|\leqslant 1$.

(iv) *The stationary process* $\{X_t\}_{t\in\mathbb{Z}}$ *has intermediate memory property if* $-0.5<d+D<0$, $-0.5<D<0$ *and* $\phi(z)\Phi(z^s)\neq 0$, *for* $|z|\leqslant 1$.

(v) *For* $|d+D|<0.5$, $|D|<0.5$, *the autocovariance function of order* h, $h\in\mathbb{Z}_{\geqslant}$, *for the process* $\{X_t\}_{t\in\mathbb{Z}}$, *is given by*

$$\gamma_X(h)=\begin{cases}\sigma_\varepsilon^2\displaystyle\sum_{\nu\in\mathbb{Z}_{\geqslant}}\gamma_Z(s\nu)\gamma_Y(h-s\nu), & \text{if } \ h=s\ell,\ \ell\in\mathbb{Z}_{\geqslant},\\ 0, & \text{if } \ h=s\ell+\zeta,\ \zeta\in A,\end{cases}\tag{2.22}$$

where $\{Z_t\}_{t\in\mathbb{Z}}$ *is a* SARFIMA$(P,D,Q)_s$ *process, where* $\{Y_t\}_{t\in\mathbb{Z}}$ *is an* ARFIMA(p,d,q) *process,* $A=\{1,\cdots,s-1\}$, $\gamma_Z(\cdot)$ *and* $\gamma_Y(\cdot)$ *are, respectively, the autocovariance function of the* $\{Z_t\}_{t\in\mathbb{Z}}$ *and* $\{Y_t\}_{t\in\mathbb{Z}}$ *processes, given by the expressions (2.37) and (2.38).*

Proof. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process, given by the expression (2.1), with seasonality $s\in\mathbb{N}$.

(i) The expression of the spectral density function of the process $\{X_t\}_{t\in\mathbb{Z}}$ follows immediately by its definition and by item (i) of Theorem 2.1,

$$f_X(w) = f_Y(w)\frac{|\theta(e^{-iw})|^2|\Theta(e^{-isw})|^2}{|\phi(e^{-iw})|^2|\Phi(e^{-isw})|^2}, \quad \text{for all } \; 0 < w \leqslant \pi, \tag{2.23}$$

where $f_Y(\cdot)$ is the spectral density function of a SARFIMA$(0,d,0)\times(0,D,0)_s$ process, given by expression (2.4). Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ is causal and invertible process. We know that $\lim\limits_{w\to 0}\cos(sw) = 1$, that is, $\cos(sw) \sim 1$, when $w \to 0$. For the expression (2.23), when $w \to 0$, one has

$$\begin{aligned} f_X(w) \;&\sim\; \frac{\sigma_\varepsilon^2}{2\pi}s^{-2D}|w-w_0|^{-2(d+D)}\left[\frac{\prod_{m=1}^{q}(1-\rho_{m,1})^2\prod_{l=1}^{Q}(1-\rho_{l,3})^2}{\prod_{\ell=1}^{p}(1-\rho_{\ell,2})^2\prod_{r=1}^{P}(1-\rho_{r,4})^2}\right] &(2.24)\\ &= C_3|w-w_0|^{-2(d+D)}, \quad \text{when} \quad w\to 0, &(2.25)\end{aligned}$$

where $w_0 = 0$ and the inverse of $\rho_{k,\iota}$, for each $k \in \{m, l, \ell, r\}$ and $\iota \in \{1,\cdots,4\}$, are the roots of the polynomials $\theta(\cdot)$, $\phi(\cdot)$, $\Theta(\cdot)$ and $\Phi(\cdot)$, with

$$C_3 = \frac{\sigma_\varepsilon^2}{2\pi}s^{-2D}\left|\frac{\theta(e^{-iw_0})\Theta(e^{-iw_0})}{\phi(e^{-iw_0})\Phi(e^{-iw_0})}\right|^2 = \frac{\sigma_\varepsilon^2}{2\pi}s^{-2D}\left[\frac{\theta(1)\Theta(1)}{\phi(1)\Phi(1)}\right]^2. \tag{2.26}$$

The expression (2.24) holds from the approximation in expression (2.8) and because the polynomials, in particular $\theta(\cdot)$, can be rewritten as

$$\theta(z) = \prod_{m=1}^{q}(1-\rho_{m,1}z), \tag{2.27}$$

where $|\rho_{m,1}| < 1$ and $1/\rho_{m,1}$ is the polynomial's root, for $m = 1,\cdots,q$. By the same manner, for each $j = 1,\cdots,\lfloor s/2\rfloor$, one has

$$\begin{aligned} f_X(\lambda+w_j) \;&=\; f_Y(\lambda+w_j)\left[\frac{|\theta(e^{-i(\lambda+w_j)})|^2|\Theta(e^{-is(\lambda+w_j)})|^2}{|\phi(e^{-i(\lambda+w_j)})|^2|\Phi(e^{-is(\lambda+w_j)})|^2}\right]\\ &\sim\; \frac{\sigma_\varepsilon^2}{2\pi}s^{-2D}\left|2\,\sin\!\left(\frac{w_j}{2}\right)\right|^{-2d}|\lambda|^{-2D}\left|\frac{\theta(e^{-iw_j})\Theta(w^{-isw_0})}{\phi(e^{-iw_j})\Phi(w^{-isw_0})}\right|^2 &(2.28)\\ &=\; C_4\,|\lambda|^{-2D}, &(2.29)\end{aligned}$$

when $\lambda \to 0$, where $w_j = \frac{2\pi j}{s}$ and

$$C_4 = \frac{\sigma_\varepsilon^2}{2\pi}s^{-2D}\left|2\,\sin\!\left(\frac{w_j}{2}\right)\right|^{-2d}\left|\frac{\theta(e^{-iw_j})\Theta(e^{-isw_0})}{\phi(e^{-iw_j})\Phi(e^{-isw_0})}\right|^2. \tag{2.30}$$

The expression (2.28) holds from the approximation in expression (2.10). By taking $\lambda = w - w_j$, in the equation (2.29), one has

$$f_X(w) \sim C_4|w-w_j|^{-2D}, \quad \text{when} \quad w \to w_j, \tag{2.31}$$

for all $j = 1,\cdots,\lfloor s/2\rfloor$, where C_4 is given by the expression (2.30).

(ii) The process $\{X_t\}_{t\in\mathbb{Z}}$ can be written as $X_t = \psi(\mathcal{B})\varepsilon_t$, where

$$\psi(z) = \frac{\theta(z)\Theta(z^s)}{\phi(z)\Phi(z^s)}(1-z)^{-d}(1-z^s)^{-D}.$$

If $d+D < 0.5$ and $D < 0.5$, item (ii) of Theorem 2.1 assures that the power series expansion of $(1-z)^{-d}(1-z^s)^{-D}$ converges for $|z| \leqslant 1$. We have that $(\phi(z)\Phi(z^s))^{-1}$ converges for $|z| \leqslant 1$ when the roots of $\phi(z)\Phi(z^s) = 0$ are outside the unit circle. Therefore, the power series $\psi(z)$ converges for all $|z| \leqslant 1$ and so the process $\{X_t\}_{t\in\mathbb{Z}}$ is stationary.

(iii) Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process, where all roots of $\phi(z)\Phi(z^s) = 0$, are outside of the unit circle, and its spectral density function is given by expression (2.21). By Definition 2.1 and by the asymptotic expression of the spectral density function given in item (i) of this theorem, the process $\{X_t\}_{t\in\mathbb{Z}}$ has long memory property when $0 < d+D < 0.5$, $0 < D < 0.5$ and all roots of $\phi(z)\Phi(z^s) = 0$ are outside of the unit circle.

(iv) By item (i) and Remark 2.1, the process $\{X_t\}_{t\in\mathbb{Z}}$ has intermediate memory property when $-0.5 < d+D < 0$ and $-0.5 < D < 0$.

(v) Let $\{X_t\}_{t\in\mathbb{Z}}$ be a causal and invertible SARFIMA$(p,d,q)\times(P,D,Q)_s$ process given by the equation (2.2). To give an expression for the autocovariance function $\gamma_X(\cdot)$, let $\{\widetilde{U}_t\}_{t\in\mathbb{Z}}$ be a causal and stationary SARMA$(P,Q)_s$ process as

$$\Phi(\mathcal{B}^s)\widetilde{U}_t = \Theta(\mathcal{B}^s)\epsilon^{\bullet}_t, \quad \text{for } t\in\mathbb{Z},$$

where $\{\epsilon^{\bullet}_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma^2_{\epsilon^{\bullet}}$. Then,

$$\widetilde{U}_t = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j \mathcal{B}^{sj}(\epsilon^{\bullet}_t) = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j \epsilon^{\bullet}_{t-sj},$$

where the coefficients ψ_j's are specified by the relationship

$$\psi(z) = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j z^{sj} = \frac{\Theta(z^s)}{\Phi(z^s)}, \text{for} |z| \leqslant 1. \tag{2.32}$$

By definition, the autocovariance function of $\{\widetilde{U}_t\}_{t\in\mathbb{Z}}$ process, denoted by $\gamma_{\widetilde{U}}(\cdot)$, is given by

$$\gamma_{\widetilde{U}}(h) = \mathrm{Cov}(\widetilde{U}_{t+h}, \widetilde{U}_t) \;=\; \sum_{j\in\mathbb{Z}_{\geqslant}}\sum_{\upsilon\in\mathbb{Z}_{\geqslant}} \psi_j\psi_\upsilon\gamma_{\epsilon^{\bullet}}(h - sj + s\upsilon),$$

for all $h\in\mathbb{Z}_{\geqslant}$. When $h - sj + s\upsilon = 0$, one has $j = \frac{h}{s} + \upsilon$. Then, $\gamma_{\epsilon^{\bullet}}(h) = \sigma^2_{\epsilon^{\bullet}}$. Hence,

$$\gamma_{\widetilde{U}}(h) = \sigma^2_{\epsilon^{\bullet}} \sum_{\upsilon\in\mathbb{Z}_{\geqslant}} \psi_{\frac{h}{s}+\upsilon}\psi_\upsilon.$$

Let $h = s\nu$, for $\nu\in\mathbb{Z}_{\geqslant}$. Then, one has

$$\gamma_{\widetilde{U}}(s\nu) = \sigma^2_{\epsilon^\bullet} \sum_{\upsilon \in \mathbb{Z}_\geqslant} \psi_{\nu+\upsilon}\psi_\upsilon. \tag{2.33}$$

If $h = s\nu + \zeta$ in equation (2.33), where $\zeta \in A$, then $\gamma_{\widetilde{U}}(s\nu + \zeta) = 0$. Therefore, the autocovariance function of the process $\{\widetilde{U}_t\}_{t\in\mathbb{Z}}$ is given by

$$\gamma_{\widetilde{U}}(h) = \begin{cases} \sigma^2_{\epsilon^\bullet} \sum_{\upsilon \in \mathbb{Z}_\geqslant} \psi_{\nu+\upsilon}\psi_\upsilon, & \text{if } h = s\nu, \ \nu \in \mathbb{Z}_\geqslant; \\ 0, & \text{if } h = s\nu + \zeta, \ \zeta \in A. \end{cases} \tag{2.34}$$

Let $\{Z_t\}_{t\in\mathbb{Z}}$ be the process given by the expression

$$Z_t = \frac{\Theta(\mathcal{B}^s)}{\Phi(\mathcal{B}^s)}\widetilde{V}_t, \text{ for all } t \in \mathbb{Z},$$

with innovation process $\{\widetilde{V}_t\}_{t\in\mathbb{Z}}$ given by $\widetilde{V}_t = (1-\mathcal{B}^s)^{-D}\epsilon^*_t$ (that is, $\{\widetilde{V}_t\}_{t\in\mathbb{Z}}$ is a SARFIMA $(0, D, 0)_s$ process with innovation process $\{\epsilon^*_t\}_{t\in\mathbb{Z}}$, a white noise process with zero mean and variance $\sigma^2_{\epsilon^*} = \mathrm{Var}(\epsilon^*_t)$). Therefore,

$$\begin{aligned} Z_t &= \frac{\Theta(\mathcal{B}^s)}{\Phi(\mathcal{B}^s)}\widetilde{V}_t = \Psi(\mathcal{B}^s)\widetilde{V}_t \\ &= \sum_{j \in \mathbb{Z}_\geqslant} \psi_j \mathcal{B}^{sj}(\widetilde{V}_t) = \sum_{j \in \mathbb{Z}_\geqslant} \psi_j \widetilde{V}_{t-sj}, \end{aligned}$$

where the coefficients ψ_j's are given by the relationship (2.32).

Since $\{Z_t\}_{t\in\mathbb{Z}}$ is a causal and stationary process, from Proposition 3.1 of Section 3, the autocovariance function of order h, $h \in \mathbb{Z}_\geqslant$, is given by

$$\gamma_Z(h) = \sigma^2_{\epsilon^*} \sum_{j \in \mathbb{Z}_\geqslant} \sum_{\upsilon \in \mathbb{Z}_\geqslant} \psi_j \psi_\upsilon \gamma_{\widetilde{V}}(h - s(j-\upsilon)). \tag{2.35}$$

Taking $m = j - \upsilon$ in the expression (2.35), one has

$$\gamma_Z(h) = \sigma^2_{\epsilon^*} \sum_{m \in \mathbb{Z}_\geqslant} \sum_{\upsilon \in \mathbb{Z}_\geqslant} \psi_{m+\upsilon} \psi_\upsilon \gamma_{\widetilde{V}}(h - sm). \tag{2.36}$$

Replacing equation (2.34) into (2.36), one has the autocovariance function of a SARFIMA$(P, D, Q)_s$ process given by

$$\gamma_Z(h) = \begin{cases} \sigma^2_{\varepsilon^*} \sum_{m \in \mathbb{Z}_\geqslant} \gamma_{\widetilde{U}}(sm)\gamma_{\widetilde{V}}(h - sm), & \text{if } h = sm, \ m \in \mathbb{Z}_\geqslant; \\ 0, & \text{if } h = sm + \zeta, \ \zeta \in A, \end{cases} \tag{2.37}$$

where $\sigma^2_{\varepsilon^*} = \sigma^2_{\epsilon^*}/\sigma^2_{\epsilon^\bullet}$, $\{\widetilde{U}_t\}_{t\in\mathbb{Z}}$ is a seasonal SARMA$(P,Q)_s$ process, $\{\widetilde{V}_t\}_{t\in\mathbb{Z}}$ is a SARFIMA$(0,D,0)_s$ process and $\{\epsilon^*_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma^2_{\epsilon^*} = 1$.

Let $\{Y_t\}_{t\in\mathbb{Z}}$ be an ARFIMA(p,d,q) process (see item (3) of Remark 2.2). From Theorem 13.2.2 item (d) of Brockwell and Davis (1991), the autocovariance function of the process $\{Y_t\}_{t\in\mathbb{Z}}$ is given by

$$\gamma_Y(h) = \sigma^2_{\varepsilon^\bullet} \sum_{j\in\mathbb{Z}_\geqslant} \gamma_{\widetilde{\widetilde{U}}}(j)\gamma_{\widetilde{\widetilde{V}}}(h-j), \tag{2.38}$$

where $\{\widetilde{\widetilde{U}}_t\}_{t\in\mathbb{Z}}$ is an ARMA(p,q) process with innovations as $\{\epsilon^{\divideontimes}_t\}_{t\in\mathbb{Z}}$, $\{\widetilde{\widetilde{V}}_t\}_{t\in\mathbb{Z}}$ is an ARFIMA$(0,d,0)$ process with innovations given by $\{\check{\epsilon}_t\}_{t\in\mathbb{Z}}$ and $\{\varepsilon^\bullet_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma^2_{\varepsilon^\bullet} = \sigma^2_{\epsilon^{\divideontimes}}/\sigma^2_{\check{\epsilon}}$.

One can obtain the autocovariance function of a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process by repeating the same method as for finding the autocovariance function of a SARFIMA $(P,D,Q)_s$ process which is given by the expression (2.22), where $\{Z_t\}_{t\in\mathbb{Z}}$ is a SARFIMA $(P,D,Q)_s$ process with innovation process $\{\varepsilon^*_t\}_{t\in\mathbb{Z}}$, $\{Y_t\}_{t\in\mathbb{Z}}$ is an ARFIMA(p,d,q) process with innovation process as $\{\varepsilon^\bullet_t\}_{t\in\mathbb{Z}}$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process with zero mean and variance $\sigma^2_\varepsilon = \sigma^2_{\varepsilon^*}/\sigma^2_{\varepsilon^\bullet}$.

□

3 Causality and Invertibility Properties

This section shows necessary and sufficient conditions for a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process to be causal and invertible. First, the following theorem presents these conditions for the causality property.

Theorem 3.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a* SARFIMA$(p,d,q)\times(P,D,Q)_s$ *process (see Definition 2.2). Suppose $d < 0.5$, $D < 0.5$ and that the equations $\phi(z)\Phi(z^s) = 0$ and $\theta(z)\Theta(z^s) = 0$ have no common zeroes. Then, $\{X_t\}_{t\in\mathbb{Z}}$ is causal if and only if $\phi(z)\Phi(z^s) \neq 0$, for all $z \in \mathbb{Z}$, such that $|z| \leqslant 1$. The coefficients $\{\psi_j\}_{j\in\mathbb{Z}_\geqslant}$ of the infinite moving average representation are given by*

$$\psi(z) = \sum_{j\in\mathbb{Z}_\geqslant} \psi_j z^j = \frac{\theta(z)\Theta(z^s)}{\phi(z)\Phi(z^s)}(1-z)^{-d}(1-z^s)^{-D}, |z| \leqslant 1. \tag{3.1}$$

Proof. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process, with zero mean, given by the expression (2.1). First one needs to prove that if $\phi(z)\Phi(z^s) \neq 0$, for all $z \in \mathbb{Z}$, such that $|z| \leqslant 1$, then the process is causal.

From Theorem 13.2.2 in Brockwell and Davis (1991), the ARFIMA(p,d,q) process is causal when $d < 0.5$ if and only if $\phi(z) \neq 0$, for all $|z| \leqslant 1$. Therefore, one can rewrite the equation (2.1), with $\mathcal{B} = z$, as

$$\Phi(z^s)(1-z^s)^D X_t = \Theta(z^s)\frac{\theta(z)(1-z)^{-d}}{\phi(z)}\varepsilon_t \Longleftrightarrow \Phi(z^s)(1-z^s)^D X_t = \Theta(z^s)Y_t, \tag{3.2}$$

for all $t \in \mathbb{Z}$, so that $\{Y_t\}_{t\in\mathbb{Z}}$ can be regarded as an ARFIMA(p,d,q) process. From Theorem 2.1 of Brietzke et al. (2005), the SARFIMA$(0,D,0)_s$ process is causal when $D < 0.5$. Thus, the equation (3.2) can be rewritten as

$$\Phi(z^s)X_t = \Theta(z^s)(1-z^s)^{-D}Y_t \Longleftrightarrow \Phi(z^s)X_t = \Theta(z^s)Z_t, \quad \text{for all} \quad t \in \mathbb{Z}, \tag{3.3}$$

where $\{Z_t\}_{t\in\mathbb{Z}}$ is a SARFIMA$(0, D, 0)_s$ process.

To prove that the process in expression (3.3) is causal, first let us assume that $\Phi(z^s) \neq 0$, for all $|z| \leqslant 1$. Therefore, $1/\Phi(z^s)$ is an analytic function and it has a power series expansion. Hence, there exists $\epsilon > 0$ such that

$$\frac{1}{\Phi(z^s)} = \sum_{j\in\mathbb{Z}_{\geqslant}} \xi_j z^j = \xi(z), \text{ for all } |z| < 1+\epsilon.$$

Since the series converges for $|z| < 1+\epsilon$, it also converges for $|z| < 1+\frac{\epsilon}{2}$. Therefore, $\lim_{j\to\infty} \xi_j \left(1+\frac{\epsilon}{2}\right) = 0$, that is, the sequence $\left\{\xi_j \left(1+\frac{\epsilon}{2}\right)\right\}_{j\in\mathbb{Z}_{\geqslant}}$ is bounded and it converges. Moreover, there exists a finite constant $K > 0$ such that,

$$\left|\xi_j \left(1+\frac{\epsilon}{2}\right)^j\right| < K, \text{ that is, } |\xi_j| < K\left(1+\frac{\epsilon}{2}\right)^{-j}, \text{ for all } j \in \mathbb{Z}_{\geqslant}.$$

In particular, one has

$$\sum_{j\in\mathbb{Z}_{\geqslant}} |\xi_j| < K \sum_{j\in\mathbb{Z}_{\geqslant}} \left(1+\frac{\epsilon}{2}\right)^{-j} < \infty \text{ and } \xi(z)\Phi(z^s) \equiv 1, \text{ for } |z| \leqslant 1.$$

From Proposition 3.1.2 in Brockwell and Davis (1991), one can apply the operator $\xi(\cdot)$ to both sides of the expression (3.3) to obtain $X_t = \xi(\mathcal{B})\Theta(\mathcal{B}^s)Z_t$. Thus one has the desired representation,

$$X_t = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j Z_{t-j}, \quad \text{for all} \quad t \in \mathbb{Z},$$

where the sequence $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ is specified by the relationship (3.1), when $p = 0 = q$ and $d = 0 = D$. Therefore, the process $\{X_t\}_{t\in\mathbb{Z}}$ is causal.

Now, we will show that if $\{X_t\}_{t\in\mathbb{Z}}$ is causal then, $\phi(z)\Phi(z^s) \neq 0$, for all $z \in \mathbb{Z}$, such that $|z| \leqslant 1$. It is enough to show that if the process given by the expression (3.3) is causal then, $\Phi(z^s) \neq 0$, for all $z \in \mathbb{Z}$, such that $|z| \leqslant 1$. Let us assume that the process is causal, i.e., $X_t = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j Z_{t-j}$, for some sequence $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ such that $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j| < \infty$. Then,

$$\Theta(\mathcal{B}^s)Z_t = \Phi(\mathcal{B}^s)X_t = \Phi(\mathcal{B}^s) \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j Z_{t-j} = \Phi(\mathcal{B}^s)\psi(\mathcal{B})Z_t. \tag{3.4}$$

Let $\eta(z)$ be $\Phi(z^s)\psi(z) = \sum_{j\in\mathbb{Z}_{\geqslant}} \eta_j z^j$, for all $|z| \leqslant 1$. The expression (3.4) can be rewritten as

$$\sum_{l=0}^{Q} \Theta_l Z_{t-sl} = \sum_{j\in\mathbb{Z}_{\geqslant}} \eta_j Z_{t-j}. \tag{3.5}$$

By taking the inner product in both sides of the equality (3.5) with $Z_{t-s\nu}$, one has

$$\sum_{l=0}^{Q} \Theta_l \mathbb{E}(Z_{t-sl}Z_{t-s\nu}) = \sum_{j\in\mathbb{Z}_{\geqslant}} \eta_j \mathbb{E}(Z_{t-j}Z_{t-s\nu}),$$

where Z_t is normally distributed with zero mean and variance equal to τ^2, for any $t \in \mathbb{Z}$. Thus,

$$\tau^2 \sum_{\nu=0}^{Q} \Theta_\nu = \tau^2 \sum_{\substack{j=0 \\ j=s\ell}}^{sQ} \eta_j, \tag{3.6}$$

with $\eta_j = 0$, for all $j > sQ$ and $j \neq s\ell$, $\ell \in \mathbb{Z}_{\geqslant}$. Considering $\upsilon = j/s$, in the right-hand side of equation (3.6), one has

$$\sum_{l=0}^{Q} \Theta_l = \sum_{\upsilon=0}^{Q} \eta_\upsilon.$$

Therefore,

$$\Theta(z^s) = \eta(z) = \Phi(z^s)\psi(z), \text{for} |z| \leqslant 1. \tag{3.7}$$

Since $\Theta(\cdot)$ and $\Phi(\cdot)$ have no common zeroes, $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j| < \infty$ and $\Theta(z^s) \neq 0$, for $|z| \leqslant 1$. From expression (3.7) one concludes that $\Phi(z^s)$ cannot be zero, for any $|z| \leqslant 1$.

□

Lemma 3.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a causal* SARFIMA$(p, d, q) \times (P, D, Q)_s$ *process (see Definition 2.2). Then, $\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j^2 < \infty$, where $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ are given by the expression* (3.1).

Proof. From the causality property, there exists a sequence $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ such that $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j| < \infty$ and

$$X_t = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j \varepsilon_{t-j}, \text{ for all } t \in \mathbb{Z},$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process and $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ are the coefficients given by the expression (3.1). Since $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j| < \infty$, then $\sum_{j\in\mathbb{Z}_{\geqslant}} |\psi_j|^2 < \infty$. Therefore, $\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j^2 < \infty$. □

The following proposition gives the mean square and the almost sure convergences for the coefficients of the infinite moving average representation for a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process. We used this result to obtain the autocovariance function for this process.

Proposition 3.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a causal and stationary* SARFIMA$(p, d, q) \times (P, D, Q)_s$ *process, given by the expression (2.1). Then, the series*

$$\psi(\mathcal{B})\varepsilon_t = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j \mathcal{B}^j(\varepsilon_t) = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j \varepsilon_{t-j}, \tag{3.8}$$

converges absolutely with probability one and in the mean square sense to the same limit.

Proof. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a causal and stationary SARFIMA$(p, d, q) \times (P, D, Q)_s$ process. One wants to show the mean square convergence of the series $\psi(\mathcal{B})\varepsilon_t$, for any $t \in \mathbb{Z}$, where the coefficients $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ are given by the expression (3.1). Let m, n be non negative integers, such that $m < n$ and define $S_m := \sum_{j=0}^{m} \psi_j \varepsilon_{t-j}$. Then,

$$\begin{aligned}\| S_n - S_m \|^2 &= \mathbb{E}\left|\sum_{\nu=0}^{n}\psi_\nu\varepsilon_{t-\nu} - \sum_{j=0}^{m}\psi_j\varepsilon_{t-j}\right|^2 = \mathbb{E}\left[\sum_{j=m+1}^{n}\psi_j\varepsilon_{t-j}\right]^2 \\ &= \mathbb{E}\left[\sum_{j=m+1}^{n}\psi_j^2\varepsilon_{t-j}^2 + \sum_{\substack{\nu,j=m+1\\ \nu\neq j}}^{n}\psi_\nu\psi_j\varepsilon_{t-\nu}\varepsilon_{t-j}\right] = \sigma_\varepsilon^2\sum_{j=m+1}^{n}\psi_j^2,\end{aligned}$$

since $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process.

It is sufficient to show that $\sum_{\nu\in\mathbb{Z}_\geqslant}\psi_\nu^2 < \infty$. Since $\{X_t\}_{t\in\mathbb{Z}}$ is a causal and stationary process, from Lemma 3.1, one has $\sum_{\nu\in\mathbb{Z}_\geqslant}\psi_\nu^2 < \infty$. Therefore, for all $\epsilon > 0$, there exists $N(\epsilon) > 0$ sufficiently large, such that $\sum_{j=m+1}^{n}\psi_j^2 < \epsilon$, for all $n > m > N(\epsilon)$. By Cauchy criterion, the series (3.8) converges in mean square sense.

From Cauchy-Schwarz inequality, $\mathbb{E}(|\varepsilon_t|^2) = \mathbb{E}(\varepsilon_t)^2 = \sigma_\varepsilon^2 < \infty$, for all $t \in \mathbb{Z}$. Thus, $\mathbb{E}(|\varepsilon_t|) < \infty$, for all $t \in \mathbb{Z}$, that is, $\sup_t \mathbb{E}(|\varepsilon_t|) < \infty$. From the Monotone Convergence Theorem, one has

$$\begin{aligned}\mathbb{E}\left(\sum_{j\in\mathbb{Z}_\geqslant}|\psi_j||\varepsilon_{t-j}|\right) &= \mathbb{E}\left(\lim_{n\to\infty}\sum_{j=0}^{n}|\psi_j||\varepsilon_{t-j}|\right) = \lim_{n\to\infty}\mathbb{E}\left(\sum_{j=0}^{n}|\psi_j||\varepsilon_{t-j}|\right) \\ &= \lim_{n\to\infty}\sum_{j=0}^{n}|\psi_j|\mathbb{E}(|\varepsilon_{t-j}|) \leqslant \lim_{n\to\infty}\sum_{j=0}^{n}|\psi_j|\sup_t\mathbb{E}(|\varepsilon_t|) = C < \infty,\end{aligned}$$

since $\sum_{j\in\mathbb{Z}_\geqslant}|\psi_j| < \infty$ and $\sup_t \mathbb{E}(|\varepsilon_t|) < \infty$. Therefore, $\sum_{j\in\mathbb{Z}_\geqslant}|\psi_j||\varepsilon_{t-j}|$ and $\sum_{j\in\mathbb{Z}_\geqslant}\psi_j\varepsilon_{t-j}$ are both finite with probability one.

Let $\mathcal{S}$ denote the mean square limit. Hence, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ sufficiently large, such that, for all $n \geqslant N$,

$$\| \mathcal{S} - \sum_{j=0}^{n}\psi_j\varepsilon_{t-j} \|^2 = \mathbb{E}\left(\left|\mathcal{S} - \sum_{j=0}^{n}\psi_j\varepsilon_{t-j}\right|^2\right) < \epsilon, \quad \text{that is,}$$

$$\lim_{n\to\infty}\mathbb{E}\left(\left|\mathcal{S} - \sum_{j=0}^{n}\psi_j\varepsilon_{t-j}\right|^2\right) = 0.$$

Therefore, from Fatou's lemma,

$$\begin{aligned}\| \mathcal{S} - \psi(\mathcal{B})\varepsilon_t \|^2 &= \mathbb{E}\left(|\mathcal{S} - \psi(\mathcal{B})\varepsilon_t|^2\right) = \mathbb{E}\left(\liminf_{n\to\infty}\left|\mathcal{S} - \sum_{j=0}^{n}\psi_j\varepsilon_{t-j}\right|^2\right) \\ &\leqslant \liminf_{n\to\infty}\mathbb{E}\left(\left|\mathcal{S} - \sum_{j=0}^{n}\psi_j\varepsilon_{t-j}\right|^2\right) = 0,\end{aligned}$$

showing that the limit $\mathcal{S}$ is equal to $\psi(\mathcal{B})\varepsilon_t$ with probability one.

□

Theorem 3.2. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a* SARFIMA$(p,d,q)\times(P,D,Q)_s$ *process (see Definition 2.2). Suppose $d>-0.5$, $D>-0.5$ and the equations $\phi(z)\Phi(z^s)=0$ and $\theta(z)\Theta(z^s)=0$ have no common zeroes. Then, $\{X_t\}_{t\in\mathbb{Z}}$ is invertible if and only if $\theta(z)\Theta(z^s)\neq 0$, for all $z\in\mathbb{Z}$, such that $|z|\leqslant 1$. The coefficients $\{\pi_j\}_{j\in\mathbb{Z}_\geqslant}$ of the infinite autoregressive representation are given by*

$$\pi(z)=\sum_{j\in\mathbb{Z}_\geqslant}\pi_j z^j=\frac{\phi(z)\Phi(z^s)}{\theta(z)\Theta(z^s)}(1-z)^d(1-z^s)^D,\ |z|\leqslant 1. \tag{3.9}$$

Proof. We omit it. It can be obtained by following the same arguments as in Theorem 3.1. □

Lemma 3.2. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be an invertible* SARFIMA$(p,d,q)\times(P,D,Q)_s$ *process (see Definition 2.2). Then, $\sum_{j\in\mathbb{Z}_\geqslant}\pi_j^2<\infty$, where $\{\pi_j\}_{j\in\mathbb{Z}_\geqslant}$ are the coefficients given by the expression (3.9).*

Proof. From the invertibility property, there exists a sequence $\{\pi_j\}_{j\in\mathbb{Z}_\geqslant}$ such that $\sum_{j\in\mathbb{Z}_\geqslant}|\pi_j|<\infty$ and

$$\varepsilon_t=\sum_{j\in\mathbb{Z}_\geqslant}\pi_j X_{t-j},\ \text{for all}\, t\in\mathbb{Z},$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process and $\{\pi_j\}_{j\in\mathbb{Z}_\geqslant}$ are the coefficients given by expression (3.9). Since $\sum_{j\in\mathbb{Z}_\geqslant}|\pi_j|<\infty$, then $\sum_{j\in\mathbb{Z}_\geqslant}|\pi_j|^2<\infty$. Therefore, $\sum_{j\in\mathbb{Z}_\geqslant}\pi_j^2<\infty$. □

Proposition 3.2. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a stationary and invertible* SARFIMA$(p,d,q)\times(P,D,Q)_s$ *process (see Definition 2.2). Then, the series*

$$\pi(\mathcal{B})X_t=\sum_{j\in\mathbb{Z}_\geqslant}\pi_j\mathcal{B}^j(X_t)=\sum_{j\in\mathbb{Z}_\geqslant}\pi_j X_{t-j}, \tag{3.10}$$

converges absolutely with probability one and in the mean square sense to the same limit.

Proof. We omit it. It can be obtained by following similarly to the proof in Proposition 3.1.

Remark 3.1. We define the function

$$S_X(w)=f_X(w)g(w),\ \text{for}\, 0<w\leqslant\pi, \tag{3.11}$$

where $f_X(\cdot)$ is the spectral density function of a SARFIMA$(0,d,0)\times(0,D,0)_s$ process, given by the expression (2.4), and $g:[-\pi,\pi]\to(0,\pi]$ is a real slowly varying function at the seasonal frequencies $w_j=\frac{2\pi j}{s}$, $j=0,1,\cdots,\lfloor s/2\rfloor$, in the Zygmund sense (see Zygmund, 1959) and it has bounded variation on $(0,\pi]\setminus\bigcup_{j=1}^{\lfloor s/2\rfloor}[w_j-\epsilon,w_j+\epsilon]$, for any $\epsilon>0$.

When $g(w)\equiv 1$, for all $w\in(-\pi,\pi]$, $S_X(\cdot)$ is the spectral density function of a SARFIMA $(0,d,0)\times(0,D,0)_s$ process.

Theorem 3.3. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a real* SARFIMA$(0,d,0)\times(0,D,0)_s$ *process with spectral density function given by the expression (3.11). Then, the asymptotic expression for the autocovariance function of $\{X_t\}_{t\in\mathbb{Z}}$, of order h, $h\in\mathbb{Z}_\geqslant$, when $h\to\infty$, is given by*

$$\gamma_X(h)=\begin{cases}\displaystyle\sum_{j=0}^{\delta}\alpha_j|h|^{2\beta_j-1}g\left(\frac{1}{h}+w_j\right)[\sin(\pi\beta_j-w_jh)+o(1)], & \text{if}\, h=s\ell,\ \ell\in\mathbb{Z}_\geqslant,\\ 0, & \text{if}\, h=s\ell+\zeta,\ \zeta\in A,\end{cases} \tag{3.12}$$

where

$$\delta = \begin{cases} \lfloor \frac{s}{2} \rfloor - 1, & \textit{if } s \textit{ is even,} \\ \\ \lfloor \frac{s}{2} \rfloor, & \textit{if } s \textit{ is odd,} \end{cases} \qquad \beta_j = \begin{cases} d + D, & \textit{if } j = 0, \\ \\ D, & \textit{if } j \neq 0, \end{cases} \tag{3.13}$$

$$\alpha_j = \begin{cases} \frac{\sigma_\varepsilon^2}{\pi}\Gamma(1-2\beta_j)s^{-2D}\left|2\,\sin\left(\frac{w_j}{2}\right)\right|^{-2d}, & \textit{if } j \neq 0, \\ \\ \frac{\sigma_\varepsilon^2}{\pi}\Gamma(1-2\beta_j)s^{-2D}, & \textit{if } j = 0, \end{cases} \tag{3.14}$$

with $w_j = \frac{2\pi j}{s}$*, for* $j = 0, 1, \cdots, \lfloor s/2 \rfloor$ *and* $g(\cdot)$ *given in Remark 3.1.*

Proof. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a real SARFIMA$(0,d,0)\times(0,D,0)_s$ process, whose spectral density function is denoted by $S_X(\cdot)$ and it is given by (3.11). Let $g(\cdot)$ denote the spectral density function of the innovation process. Therefore, by definition of the autocovariance function of a SARFIMA$(0,d,0)\times(0,D,0)_s$ process, $\gamma_X(h) = 0$, for $h = s\ell+\zeta$, with $\zeta \in A$. The autocovariance function of order $h = s\ell$, $\ell \in \mathbb{Z}_\geqslant$ and s even, is given by

$$\begin{aligned} \gamma_X(h) &= \int_{-\pi}^{\pi} S_X(\lambda)\cos(\lambda h)d\lambda = 2\int_0^{\pi} S_X(\lambda)\cos(\lambda h)d\lambda \\ &= 2\sum_{j=0}^{\lfloor s/2\rfloor-1}\int_{w_j}^{w_{j+1}} S_X(\lambda)\cos(\lambda h)d\lambda = 2\sum_{j=0}^{\lfloor s/2\rfloor-1}\int_{w_0}^{w_1} S_X(w+w_j)\cos((w+w_j)h)dw \\ &= 2\sum_{j=0}^{\lfloor s/2\rfloor-1}\left[\cos(w_j h)\int_{w_0}^{w_1} S_X(w+w_j)\cos(wh)dw + \sin(w_j h)\int_{w_0}^{w_1} S_X(w+w_j)\sin(wh)dw\right], \end{aligned} \tag{3.15}$$

where $w_j = \frac{2\pi j}{s}$, for $j = 0, 1, \cdots, \lfloor s/2 \rfloor$. The expression (3.15) follows immediately by considering $\lambda = w + w_j$.

From expression (2.4), from the asymptotic behavior of the spectral density function of a SARFIMA$(0,d,0)\times(0,D,0)_s$ process (see item (i) of Theorem 2.1), and from Lemma 2 in Giraitis and Leipus (1995) with $l(\cdot) \equiv g(\cdot)$, one has

$$\gamma_X(h) = \frac{\sigma_\varepsilon^2}{\pi} s^{-2D} \Bigg[\cos(w_0 h)|h|^{2(d+D)-1} g\left(\frac{1}{h} + w_0\right) \Gamma(1-2(d+D)) \times [\sin(\pi(d+D)) + o(1)]$$

$$- \sin(w_0 h)|h|^{2(d+D)-1} g\left(\frac{1}{h} + w_0\right) \times \Gamma(1-2(d+D))[\cos(\pi(d+D)) + o(1)] \Bigg]$$

$$+ \frac{\sigma_\varepsilon^2}{\pi} s^{-2D} \sum_{j=1}^{\lfloor s/2 \rfloor - 1} \left| 2 \sin\left(\frac{w_j}{2}\right)\right|^{-2d} \times \Bigg[\cos(w_j h)|h|^{2D-1} g\left(\frac{1}{h} + w_j\right) \Gamma(1-2D)$$

$$\times [\sin(\pi D) + o(1)] - \sin(w_j h)|h|^{2D-1} g\left(\frac{1}{h} + w_j\right) \Gamma(1-2D)[\cos(\pi D) + o(1)] \Bigg]$$

$$= \frac{\sigma_\varepsilon^2}{\pi} s^{-2D} |h|^{2(d+D)-1} g\left(\frac{1}{h} + w_0\right) \Gamma(1-2(d+D))[\sin(\pi(d+D) - w_0 h) + o(1)]$$

$$+ \frac{\sigma_\varepsilon^2}{\pi} s^{-2D} \sum_{j=1}^{\lfloor s/2 \rfloor - 1} \left| 2 \sin\left(\frac{w_j}{2}\right)\right|^{-2d} |h|^{2D-1} g\left(\frac{1}{h} + w_j\right) \Gamma(1-2D)$$

$$\times [\sin(\pi D - w_j h) + o(1)], \tag{3.16}$$

when $h \to \infty$.

The autocovariance function of order $h = s\ell$, $\ell \in \mathbb{Z}_{\geqslant}$ and s odd, is given by

$$\gamma_X(h) = 2\int_0^\pi S_X(\lambda)\cos(\lambda h) d\lambda = 2 \sum_{j=0}^{\lfloor s/2 \rfloor - 1} \int_{w_j}^{w_{j+1}} S_X(\lambda)\cos(\lambda h) d\lambda + 2\int_{w_{\lfloor s/2 \rfloor - 1}}^{\pi} S_X(\lambda)\cos(\lambda h) d\lambda$$

$$= 2 \sum_{j=0}^{\lfloor s/2 \rfloor - 1} \int_{w_0}^{w_1} S_X(w + w_j)\cos((w + w_j)h) dw + 2\int_{w_0}^{\frac{w_1}{2}} S_X(w + w_{[s/2]})\cos((w + w_{\lfloor s/2 \rfloor})h) du \tag{3.17}$$

$$= 2 \sum_{j=0}^{\lfloor s/2 \rfloor - 1} \Bigg[\cos(w_j h) \int_{w_0}^{w_1} S_X(w + w_j)\cos(wh) dw - \sin(w_j h) \int_{w_0}^{w_1} S_X(w + w_j)\cos(wh) dw \Bigg]$$

$$+ 2\cos(w_{\lfloor s/2 \rfloor} h) \int_{w_0}^{\frac{w_1}{2}} S_X(w + w_{\lfloor s/2 \rfloor})\cos(wh) dw - 2\sin(w_{\lfloor s/2 \rfloor} h) \int_{w_0}^{\frac{w_1}{2}} S_X(w + w_{\lfloor s/2 \rfloor})\sin(wh) dw, \tag{3.18}$$

with $w_j = \frac{2\pi j}{s}$, for $j = 0, 1, \cdots, \lfloor s/2 \rfloor$. The expression (3.17) follows immediately by considering $\lambda = w + w_j$.

From expression (2.4), from the asymptotic behavior of the spectral density function of a SARFIMA$(0, d, 0) \times (0, D, 0)_s$ process (see item (i) of Theorem 2.1), and from Lemma 2 in Giraitis and Leipus (1995) with $l(\cdot) \equiv g(\cdot)$, one has

$$\gamma_X(h) = \frac{\sigma_\varepsilon^2}{\pi} s^{-2D}\Big[\cos(w_0 h)|h|^{2(d+D)-1} g\left(\frac{1}{h}+w_0\right)\Gamma(1-2(d+D))[\sin(\pi(d+D))+o(1)]$$

$$-\sin(w_0 h)|h|^{2(d+D)-1} g\left(\frac{1}{h}+w_0\right)\Gamma(1-2(d+D))[\cos(\pi(d+D))+o(1)]\Big]$$

$$+\frac{\sigma_\varepsilon^2}{\pi} s^{-2D}\sum_{j=1}^{\lfloor s/2\rfloor -1}\left|2\,\sin\Big(\frac{w_j}{2}\Big)\right|^{-2d}\Big[\cos(w_j h)|h|^{2D-1} g\left(\frac{1}{h}+w_j\right)\Gamma(1-2D)$$

$$\times[\sin(\pi D)+o(1)] - \sin(w_j h)|h|^{2D-1} g\left(\frac{1}{h}+w_j\right)\Gamma(1-2D)[\cos(\pi D)+o(1)]\Big]$$

$$+\frac{\sigma_\varepsilon^2}{\pi} s^{-2D}\left|2\,\sin\Big(\frac{w_{\lfloor s/2\rfloor}}{2}\Big)\right|^{-2d}\Big[\cos(w_{\lfloor s/2\rfloor} h)|h|^{2D-1} g\left(\frac{1}{h}+w_{\lfloor s/2\rfloor}\right)\Gamma(1-2D)$$

$$\times[\sin(\pi D)+o(1)] - \sin(w_{\lfloor s/2\rfloor} h)|h|^{2D-1} g\left(\frac{1}{h}+w_{\lfloor s/2\rfloor}\right)\Gamma(1-2D)[\cos(\pi D)+o(1)]\Big]$$

$$= \frac{\sigma_\varepsilon^2}{\pi} s^{-2D}|h|^{2(d+D)-1} g\left(\frac{1}{h}+w_0\right)\Gamma(1-2(d+D))[\sin(\pi(d+D)-w_0 h)+o(1)]$$

$$+\frac{\sigma_\varepsilon^2}{\pi} s^{-2D}\sum_{j=1}^{\lfloor s/2\rfloor}\left|2\,\sin\Big(\frac{w_j}{2}\Big)\right|^{-2d}|h|^{2D-1} g\left(\frac{1}{h}+w_j\right)\Gamma(1-2D)[\sin(\pi D - w_j h)+o(1)] \quad (3.19)$$

when $h \to \infty$.

Comparing equations (3.16) and (3.19), one has the asymptotic expression of the autocovariance function of $\{X_t\}_{t\in\mathbb{Z}}$ of order h, for $h \in \mathbb{Z}_{\geqslant}$, when $h \to \infty$, given by expressions (3.12)-(3.14). □

The following proposition presents the asymptotic expression for the autocovariance function of a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process.

Proposition 3.3. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a real* SARFIMA$(p,d,q)\times(P,D,Q)_s$ *process, causal and invertible, given by the expression (2.1), with p, P, q, Q and s finite and non negative integers. Then, the asymptotic expression of the autocovariance function of $\{X_t\}_{t\in\mathbb{Z}}$ of order h, $h \in \mathbb{Z}_{\geqslant}$, when $h \to \infty$, is given by*

$$\gamma_X(h) = \begin{cases} \displaystyle\sum_{j=0}^{\delta} \alpha_j |h|^{2\beta_j - 1} g(w_j)[\sin(\pi\beta_j - w_j h) + o(1)], & \text{if } h = s\ell,\ \ell \in \mathbb{Z}_{\geqslant}, \\ 0, & \text{if } h = s\ell + \zeta,\ \zeta \in A, \end{cases} \quad (3.20)$$

where α_j, β_j and δ are given, respectively, by equations (3.14) and (3.13), $w_j = \frac{2\pi j}{s}$, for $j = 0, 1, \cdots, \lfloor s/2\rfloor$ and

$$g(w) := \left|\frac{\theta(e^{-iw})\Theta(e^{-isw})}{\phi(e^{-iw})\Phi(e^{-isw})}\right|^2. \quad (3.21)$$

PROOF. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a SARFIMA$(p,d,q)\times(P,D,Q)_s$ process, with zero mean, given by the expression (2.1). The real function $g(\cdot)$, defined by the expression (3.21), has bounded derivative. One needs to proof that $g(\cdot)$ is a slowly varying function, for all $0 < w \leqslant \pi$, such that $g(w) \neq 0$. By Binghan et al. (1987), a sufficient condition for a function $f(\cdot)$ be slowly varying at $w = 0$, in Zygmund's sense, is the existence of its derivative such that

$$\lim_{w\to 0} \frac{wf'(w)}{f(w)} = 0.$$

Since the process is causal and invertible (see Theorems 3.1 and 3.2, respectively), one has $g(w) \neq 0$, for all $0 < w \leqslant \pi$.

Therefore,

$$\ln(g(w)) = \ln|\theta(e^{-iw})|^2 + \ln|\Theta(e^{-isw})|^2 - \ln|\phi(e^{-iw})|^2 - \ln|\Phi(e^{-isw})|^2. \tag{3.22}$$

By definition,

$$g'(w) = g(w)[\ln(g(w))]'. \tag{3.23}$$

From equation (2.27), one has $\ln|\theta(e^{-iw})|^2$ equal to

$$\ln\left|\prod_{m=1}^{q}\left(1-\rho_{m,1}e^{-iw}\right)\right|^2 = \sum_{m=1}^{q}\ln\left(1-2\rho_{m,1}\cos(w)+\rho_{m,1}^2\right). \tag{3.24}$$

Therefore,

$$[\ln|\theta(e^{-iw})|^2]' = \sum_{m=1}^{q}\frac{2\rho_{m,1}\sin(w)}{\left(1-2\rho_{m,1}\cos(w)+\rho_{m,1}^2\right)}, \quad \text{for} \quad |\rho_{m,1}|<1\cdot$$

Similarly, one can rewrite the others polynomials and, by equation (3.21), the expression (3.23) can be given by

$$\begin{aligned} g'(w) \;=\; & \left|\frac{\theta(e^{-iw})\Theta(e^{-isw})}{\phi(e^{-iw})\Phi(e^{-isw})}\right|^2 \times \left[\sum_{m=1}^{q}\frac{2\rho_{m,1}\sin(w)}{\left(1-2\rho_{m,1}\cos(w)+\rho_{m,1}^2\right)} + \sum_{l=1}^{Q}\frac{2s\rho_{l,3}\sin(sw)}{\left(1-2\rho_{l,3}\cos(sw)+\rho_{l,3}^2\right)}\right. \\ & \left. -\sum_{\ell=1}^{p}\frac{2\rho_{\ell,2}\sin(w)}{\left(1-2\rho_{\ell,2}\cos(w)+\rho_{\ell,2}^2\right)} - \sum_{r=1}^{P}\frac{2s\rho_{r,4}\sin(sw)}{\left(1-2\rho_{r,4}\cos(sw)+\rho_{r,4}^2\right)}\right]. \end{aligned} \tag{3.25}$$

From equation (3.23), to verify if $g(\cdot)$ is a slowly varying function for all $w \in (0,\pi]$, such that $g(w) \neq 0$, one needs to prove that $\lim_{w\to 0} w[\ln(g(w))]' = 0$. First, one observes that

$$\lim_{w\to 0}\sum_{r=1}^{u}\frac{2\rho_{r,\iota}\sin(w)}{\left(1-2\rho_{r,\iota}\cos(w)+\rho_{r,\iota}^2\right)} = 0 = \lim_{w\to\pi}\sum_{r=1}^{u}\frac{2\rho_{r,\iota}\sin(w)}{\left(1-2\rho_{r,\iota}\cos(w)+\rho_{r,\iota}^2\right)}, \tag{3.26}$$

where u is fixed and $\left(1-\rho_{r,\iota}\right)^2 \neq 0$, for $|\rho_{r,\iota}| < 1$ and $\iota \in \{1,\cdots,4\}$. The left-hand side of the equality in expression (3.26) implies that the four terms inside the brackets in expression (3.25) go to zero. This shows that $g(w)$ is a slowly varying function at $w = 0$. Now, one needs to verify if $g(w)$ is a slowly varying function at $w \in (0,\pi]$. For this, one needs to show that

$$\lim_{w\to\pi} w\,[\ln(g(w))]' = \lim_{w\to\pi}\frac{wg'(w)}{g(w)} = 0. \tag{3.27}$$

Note also, in a similar way, that the right-hand side of the equality in expression (3.26) implies $g(\cdot)$ is a slowly varying function at $w \in (0, \pi]$, such that $g(w) \neq 0$.

One needs to verify if $g(\cdot)$ is of bounded variation at $(0, \pi]$. Let $0 < x_0 < x_1 < \cdots < x_k \leqslant \pi$ be a partition of the interval $(0, \pi]$. Then,

$$\sum_{j=1}^{k} |g(x_j) - g(x_{j-1})| \leqslant \sum_{j=1}^{k} (|g(x_j)| + |g(x_{j-1})|) < \infty, \tag{3.28}$$

since $g(w)$ is bounded, for all $w \in (0, \pi]$. Thus, $g(\cdot)$ is of bounded variation at the interval $(0, \pi]$.

As $\lim_{h\to\infty} g\left(\frac{1}{h} + w\right) = g(w)$, from Theorem 3.3, one has the asymptotic expression of the autocovariance function for a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process given by expression (3.20), where $g(\cdot)$ is given by (3.21).

□

4 Ergodicity Property

In this section we analyze the ergodicity of a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process.

Theorem 4.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a fractionally integrated ARMA process, as in item (3) of Remark 2.2, with mean $\mu = 0$, where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is a white noise process. If $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary and causal process then, it is ergodic.*

Proof. Let $\{X_t\}_{t\in\mathbb{Z}}$ be a fractionally integrated ARMA process with $\mu = 0$. From causality one has

$$X_t = \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j \varepsilon_{j-t}, \text{ for all } t \in \mathbb{Z},$$

where $\{\psi_j\}_{j\in\mathbb{Z}_{\geqslant}}$ are the infinite moving average representation coefficients of the process.

From Durrett (1996) (see Theorem 1.3 in Chapter 6), one needs to prove that $\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j^2 < \infty$. Since $\{X_t\}_{t\in\mathbb{Z}}$ is a causal and stationary process, one has

$$\gamma_X(0) = \mathbb{E}(X_t^2) = \sigma_\varepsilon^2 \sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j^2 < \infty,$$

that is, $\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j^2 < \infty$. Therefore, the process $\{X_t\}_{t\in\mathbb{Z}}$ is ergodic.

□

Corollary 4.1 presents the ergodicity for a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process.

Corollary 4.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be a causal and stationary* SARFIMA$(p, d, q) \times (P, D, Q)_s$ *process (see Definition 2.2). Then, $\{X_t\}_{t\in\mathbb{Z}}$ is an ergodic process.*

Remark 4.1. Since the series $\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j < \infty$ and the coefficients ψ_j's are positive real numbers then, $\sum_{j\in\mathbb{Z}_{\geqslant}} \psi_j^2 < \infty$. Therefore, a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process is stationary and ergodic.

Remark 4.2. *The ergodicity of a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process is very important for the purpose of Monte Carlos's simulation. For an extensive Monte Carlo's simulation study, where several estimation procedures are presented for all parameters of a SARFIMA$(p, d, q) \times (P, D, Q)_s$ process, see Bisognin and Lopes (2008). This companion paper, to be published elsewhere, also presents forecasting and an interesting application for this process.*

5 Conclusions

In this paper we give several theoretical properties of SARFIMA$(p, d, q) \times (P, D, Q)_s$ processes. We show the spectral density function and its behavior near the seasonal frequencies, the stationarity, the intermediate and long memory properties and the autocovariance function and its asymptotic expression. We also analyze the ergodicity, causality and invertibility conditions for these processes.

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