Generalized Langevin equation driven by Lévy processes: A probabilistic, numerical and time series based approach.

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Abstract

Lévy processes have been widely used to model a large variety of stochastic processes under anomalous diffusion. In this note we show that Lévy processes play an important role in the study of the Generalized Langevin Equation (GLE). Solution to GLE is proposed using stochastic integration in the sense of convergence in probability. Properties of the solution processes are obtained and numerical methods for stochastic integration are developed and applied to examples. Time series methods are applied to obtain estimation formulas to parameters related to the solution process. A Monte Carlo simulation study shows the estimation of the memory function parameter. We also estimate the stability index parameter when the noise is a Lévy process.

Keywords: Lévy processes, Langevin equation, anomalous diffusion, time series.

1 Introduction

The use of Lévy processes and stable distributions to model complex systems constitutes a rich area of research. In recent decades, the interest in nonnormal probability models has grown considerably in several fields of science. Applications can be found in laser cooling [1], turbulent flow [2], channeling in crystals [3], evolution of stock prices [4, 5, 6], protein diffusion structures [7], optimization and search problems [8], human travel [9], etc... Frequently in these models, solutions to stochastic differential equations are searched and these solutions can be cast as stochastic integrals. In this context, GLE (Generalized Langevin Equation) driven by Lévy processes may arise as a model to such phenomena. The main difficulty relies on the fact that Itô's formula cannot be used, since the stochastic process does not have finite second moment. To overcome this problem, we propose a solution to the GLE

(Generalized Langevin Equation) using stochastic integration in the sense of convergence in probability.

In Statistical Mechanics, the GLE

$$V'(t) = -\int_0^t \Gamma(t-s)V(s)ds + \eta(t),\ V(0) = V_0 \tag{1.1}$$

is used to study the movement of a Brownian particle immersed in a fluid and subject to random collisions with molecules of the fluid [10, 11]. In this equation, $V = (V(t),\ t \geq 0)$ and $\eta = (\eta(t),\ t \geq 0)$ are stochastic processes which represent respectively the velocity of the particle and the random force acting on it caused by the collisions. The process η is called *fluctuation* or *noise* and Γ, called *memory function*, is a deterministic function.

When the memory function is a δ Dirac function, i.e. $\Gamma(t) = \gamma\delta(t)$ where γ is a constant and η is the white noise, the GLE becomes the classical Langevin equation

$$V'(t) = -\gamma V(t) + \eta(t), \quad V(0) = V_0. \tag{1.2}$$

A formal way to study equation (1.2) makes use of Laplace transforms, as is done in [10] or [11]. Under this approach, the solution is given by the process

$$V(t) = V_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}\eta(s)ds. \tag{1.3}$$

Alternatively, a rigorous mathematical study of the classical Langevin equation requires the use of Itô's stochastic calculus [12]. In this case, V satisfies the stochastic differential equation

$$dV(t) = -\gamma V(t)dt + \beta dW(t), \quad V(0) = V_0, \tag{1.4}$$

where $W = (W(t),\ t \geq 0)$ is the Wiener process, also called Standard Brownian motion. Equation (1.4) is just a differential representation of the integral equation

$$V(t) = V_0 + \int_0^t -\gamma V(s)ds + \int_0^t \beta dW(s), \tag{1.5}$$

where in the second integral we have Itô's integral of the white noise, $\eta(t)dt = \beta dW(t)$. Equation (1.4), or equivalently (1.5), can be solved applying Itô's formula and the solution process, called Ornstein-Uhlenbeck process, is given by

$$V(t) = V_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)}dW(s). \tag{1.6}$$

Assuming that all the processes are of second order, that is, they have finite quadratic mean, Kannan in [13] studied a subclass of GLE. Solutions were obtained using Bochner integral [14] and the notion of derivative in Hilbert spaces. It was shown that any *mean square solution*, $V = (V(t),\ t \geq 0)$, of GLE has the form

$$V(t) = V_0\rho(t) + \int_0^t \rho(t-s)\eta(s)ds, \tag{1.7}$$

where ρ is a deterministic function satisfying the Volterra integro-differential equation

$$\rho'(t) = -\int_0^t \Gamma(t-s)\rho(s)ds, \quad \rho(0) = 1. \tag{1.8}$$

Note that when $\Gamma(t-s) = -\gamma\delta(t-s)$, the integro-differential equation (1.8) leads to $\rho(t) = e^{-\gamma t}$ and Kannan's representation (1.7) corresponds to equation (1.3), equivalently, to the Ornstein-Uhlenbeck process (1.6), with $\eta(s)ds = \beta dW(s)$.

Dropping the hypothesis of finite quadratic mean, neither Itô's stochastic calculus nor Kannan's approach can be applied. Nevertheless, (1.3), (1.6), (1.7) and (1.8) suggest that solutions weaker than *mean square solution* can be derived. The main goal of this work is to handle this task by using a weaker concept of stochastic integration, called stochastic integration in the sense of convergence in probability. Our approach is presented in section 2 and more theoretical details are presented in the Appendix.

Potential applications of our approach can be found in the GLE based modeling of anomalous diffusions, a sort of phenomenon observed in some physical systems. That is, if

$$X(t) = \int_0^t V(s)ds \tag{1.9}$$

is the position of a particle with velocity $V = (V(t),\ t \geq 0)$, then the quadratic mean displacement $\mathbb{E}[X^2(t)]$ does not grow linearly as time $t \to \infty$. See, for instance, [16, 17, 18, 19], where GLE is used to study anomalous diffusions. In this case, to handle processes with diverging moments, we introduced in [17] a generalization of the *anomalous diffusion index* that is suitable to use in cases of Lévy motions. For example, it is possible to show that, under mild conditions, the moment $m(t) = E\left[V(t)\right]$ is given by the convolution equation

$$\mu\rho(t) = m'(t) + m\int_0^t \Gamma(t-s)\rho(s)ds, \tag{1.10}$$

where $\mu = E\left[L(1)\right]$ and $m = E\left[V_0\right]$, see [20] for details. We just briefly mention that up to now we were not able to overcome some mathematical dificulties trying to perform derivations to obtain expressions for the moments of fractional order. Extensions of this work related to this point and also to Lévy flights subject to external force fields are expected. We refer the reader to the paper [21] for insights.

The paper is organized as follows. In Section 2 we present the solution process to GLE and two examples that will support the following sections. In Section 3 we take example 1 from Section 2 and perform a numerical illustration based on it. Section 4 presents the estimation procedures used in this work for both, the memory function parameter from example 2 in Section 2 and the stability index parameter estimations. Section 5 gives a Monte Carlo simulation study presenting the estimation of the memory function and stability index parameters. Section 6 presents the conclusions while in the Appendix we derive the solution in the sense of convergence in probability to the GLE and the characteristic function of S when $(L(t);\ t \geq 0)$ is a Lévy process.

2 Solution to GLE

Consider the GLE (1.1) where the noise process η does not necessarily has finite second moment. Clearly, the relation $\eta(s)ds = \beta dW(s)$ may no longer hold and the Wiener process W needs to be replaced by a process that is not necessarily Gaussian or of second order. This leads us to consider a Lévy process $L = (L(t); \quad t \geq 0)$ and model the noise as $\eta(s)ds = dL(s)$. Motivated by the previous representation formulas (1.3), (1.6), (1.7) and (1.8), we establish the following definition:

Main definition: We say that the pair (V, ρ) represents a *solution in the sense of convergence in probability* to the GLE if $V = (V(t); \ t \geq 0)$ is a stochastic process of the form

$$V(t) = V_0\rho(t) + \int_0^t \rho(t-s)dL(s) \tag{2.1}$$

and $\rho = (\rho(t); \ t \geq 0)$ is a deterministic function that satisfies the deterministic integro-differential equation

$$\rho'(t) = -\int_0^t \Gamma(t-s)\rho(s)ds, \quad \rho(0) = 1. \tag{2.2}$$

The integrator $L = (L(t), \ t \geq 0)$ in equation (2.1) is a Lévy process and the stochastic integral is taken in the sense of convergence in probability. In the Appendix we show how this integral is constructed. We will call ρ a *resolvent function.*

Recall that the characteristic function $f(t, \lambda)$ of $L(t)$ is

$$f(t, \lambda) = \exp\{(-t\psi(\lambda))\} \quad \text{for all } \lambda \text{ real}, \tag{2.3}$$

where the function $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ is the *characteristic exponent* of $L(1)$ [15]. This formula will be used in the Appendix to obtain the characteristic function of the stochastic integral appearing in equation (2.1) (Proposition A.1). It will be also used in section 3 to obtain numerical simulations to the theoretical Probability Distribution Function (PDF) of the process V (Figures 1 and 2).

Example 1: The memory function

$$\Gamma(t) = \int_0^\infty f(x)cos(xt)dx \tag{2.4}$$

appeared in [22], being f given by

$$f(x) = \begin{cases} \dfrac{2\sigma}{\pi}\left(\dfrac{x}{x_0}\right)^{1-\theta} & x \leq x_0 \\ \\ 0 & x > x_0 \end{cases} . \tag{2.5}$$

In the next section, we will solve numerically the integro-differential equation (2.2) for ρ in this example with $\theta = 1$, $\sigma = 0.1$ and $x_0 = 2$ and then, this solution will be used to generate the process V.

Example 2: Consider $\Gamma(t-s) = \gamma\delta(t-s)$, where δ is the δ Dirac function. So, (2.2) leads to the pair

$$\rho(t) = e^{-\gamma t} \quad \text{and} \quad V(t) = V_0 e^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} dL(s). \tag{2.6}$$

Stochastic processes as in this example has been called *Ornstein-Uhlenbeck Type Processes* and used in the context of stochastic volatility models [23].

This example will be used in Sections 4 and 5 to access some numerical results. We will present a Monte Carlo simulation study showing the estimation of the memory function γ parameter. The goal is to estimate the parameter γ by time series generated from the solution process (2.6) when the noise $(L(t);\ t \geq 0)$ is either the Brownian motion or a Lévy process. By using a two step algorithm we also analyze the estimation of the stability index parameter α when the noise is a Lévy process.

3 Numerical Illustration

In this section we use the memory function (2.4) introduced in example 1 to illustrate numerically our theoretical approach. That memory function comes from a concrete case considered in [22]. Based on the construction of the stochastic integral described in the appendix, we perform a numerical integration of the stochastic integral S defined by equation (A.3) and check numerically the convergence in probability. Our conclusions are summarized in figures 1 and 2.

As a first step, we approximate the integral S by the Stieltjes sum

$$S \approx \sum_{k=1}^{N} \rho(t_k) \left(L(t_k) - L(t_{k-1}) \right), \tag{3.1}$$

as is shown in (A.2). In this discretization the interval $[0, t]$ is subdivided into N equal pieces each of size Δt. In order to proceed with the numerical integration, we must consider that in this approximation the Lévy process is constant in each interval Δt, so we must rescale $\Delta L\,(t_k)$ to reflect this numerical assumption. And this is justified by the scaling property of Lévy process

$$\Delta L_k \stackrel{D}{=} L\,(t_k)\,(\Delta t)^{\frac{1}{\alpha}}\,, \tag{3.2}$$

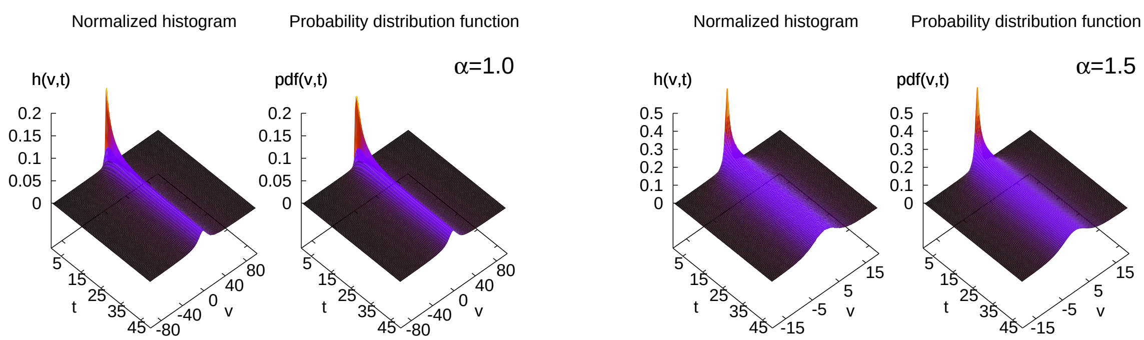
where $\stackrel{D}{=}$ means equality in distribution and α is the Lévy alpha-stable exponent. These Lévy numbers were generated using the algorithm described in [24].

Next, for the memory function (2.4) of the example 1, where f is given by (2.5), we solve numerically the integro-differential equation (2.2) for ρ with $\theta = 1$, $\sigma = 0.1$ and $x_0 = 2$. This solution is then used to generate the process V in (2.1) applying the numerical integration described above for equation (3.1).

We use a Lévy process with symmetric α-stable distribution. In this case, the characteristic function of the stochastic integral S is given by (A.4) with

$$\psi(\lambda) = |\lambda|^{\alpha}\,. \tag{3.3}$$

This characteristic function is then used to derive the Probability Distribution Function of the process V (PDF) by numerical Fourier transforms. We compare this PDF with a normalized histogram generated by the stochastic integration of equation (3.1) with 10^5 different trajectories. Without loss of generality, we choose $V(0) = 0$, which means $\mathrm{PDF}(0, V) = \delta(V)$ as boundary condition. The results are summarized in Figures 1 and 2. In Figures 1(a) and 1(b) we have the normalized histogram at left and the PDF at right, both as a function of t and V, with $\alpha = 1.0$ in Figure 1(a) and $\alpha = 1.5$ in Figure 1(b). Note the topological similarity between the histograms and the PDFs. Finally, in Figure 2 we have two contour plots, at left we have $\alpha = 1.0$ and at right we have $\alpha = 1.5$. In this figure, the fluctuating lines belong to the histograms while the smooth line belong to the PDFs. Our simulations indicate excellent numerical agreement between them.



(a) (b)

Figure 1: Normalized histogram (h(v,t) on left) and PDF (pdf(v,t) on right) as a function of t and V with: $\alpha = 1.0$ figure 1(a) and $\alpha = 1.5$ figure 1(b).

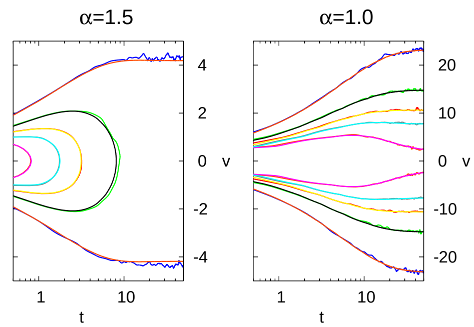


Figure 2: Contour plots for the normalized histograms and PDFs as a function of time t and V with $\alpha = 1.0$ (right) and $\alpha = 1.5$ (left).

4 Estimation Methods

Consider the model in example 2 where the γ memory function parameter is supposed to be unknown. In Section 4.1 we present three estimation methods for this parameter, two of them derived from the sample autocorrelation function of the process $V = (V(t);\ t \geq 0)$ and proposed by [25] and the ordinary least squares estimator as the third one. We also present in Section 4.2 an estimator for the stability index parameter α, when the noise in the solution to GLE is a α-stable Lévy process.

4.1 Estimation for γ

Let $(V(t);\ t \in [0,T])$ be the solution process given by expression (2.1) in the interval $[0,T]$. Consider > 0 and let V be the discrete observations of the process $(V(t);\ t \in [0,T])$ such that

$$V = \{V_0,\, V_h,\, V_{2h}, \cdots,\, V_{kh}, \cdots,\, V_T\} = \left\{V_0,\, V_1,\, \cdots,\, V_{\frac{T}{h}}\right\} \equiv \{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}. \tag{4.1}$$

The autocorrelation function of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$ is given by

$$\rho_V(k) = e^{-\gamma |k| h}. \tag{4.2}$$

Hence, from the expression (4.2), an estimator for the parameter γ proposed by [25] is given by

$$\hat{\gamma_1} = -\frac{\ln(|\hat{\rho}_V(1)|)}{h}, \tag{4.3}$$

where $\hat{\rho}_V(\cdot)$ is the sample autocorrelation of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$.

Another estimator for γ, proposed by [25], and also based on the sample autocorrelation function of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$ is given by

$$\hat{\gamma_2} = arg \min_{\gamma} \sum_{k=1}^{m^*} \left(\hat{\rho}_V(k) - e^{-\gamma k h}\right)^2, \text{ for some } m^* \text{ such that } m^* < n, \tag{4.4}$$

where $\hat{\rho}_V(\cdot)$ is the sample autocorrelation of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$ and $n = \lfloor \frac{T}{h} \rfloor$ is the sample size.

From the ordinary least squares method [26] one can obtain

$$\hat{\gamma_3} = -\frac{1}{h} \ln \left(\frac{\sum_{k=1}^{\lfloor \frac{T}{h} \rfloor} \left(V_{kh} - \overline{V}\right)\left(V_{(k-1)h} - \overline{Y}\right)}{\sum_{k=1}^{\lfloor \frac{T}{h} \rfloor} \left(V_{(k-1)h} - \overline{Y}\right)^2} \right), \tag{4.5}$$

where $\overline{V} = \frac{1}{\lfloor \frac{T}{h} \rfloor} \sum_{t=1}^{\lfloor \frac{T}{h} \rfloor} V_{th}$ and $\overline{Y} = \frac{1}{\lfloor \frac{T}{h} \rfloor} \sum_{t=1}^{\lfloor \frac{T}{h} \rfloor} V_{(t-1)h}$.

4.2 Estimation for α

Let $(V(t);\ t \in [0,T])$ be the solution process given by the expression (2.1) where now the noise $(L(t);\ t \in [0,T])$ is a Lévy flight process with α as the stability index parameter. We propose an estimator for the α stability index parameter by the following two-step algorithm:

Step 1: Estimate the parameter γ from any estimation procedure given in Section 4.1, obtaining $\hat{\gamma}$.

Step 2: Consider the residuals of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$. Then, to estimate α [27] proposed the following estimator

$$\hat{\alpha} = \frac{\ln\left(\frac{\ln\left(|\hat{\varphi}_V(t_1)|\right)}{\ln\left(|\hat{\varphi}_V(t_2)|\right)}\right)}{\ln\left(\frac{t_1}{t_2}\right)}, \tag{4.6}$$

where $\hat{\varphi}_V(\cdot)$ is the empirical characteristic function of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$ given by

$$\hat{\varphi}_V(t) = \frac{1}{n} \sum_{t=1}^{n} e^{itV_t}, \tag{4.7}$$

t_1 and t_2 are such that $0 < t_1 < t_2$ and n is the sample size.

5 Monte Carlo Simulations

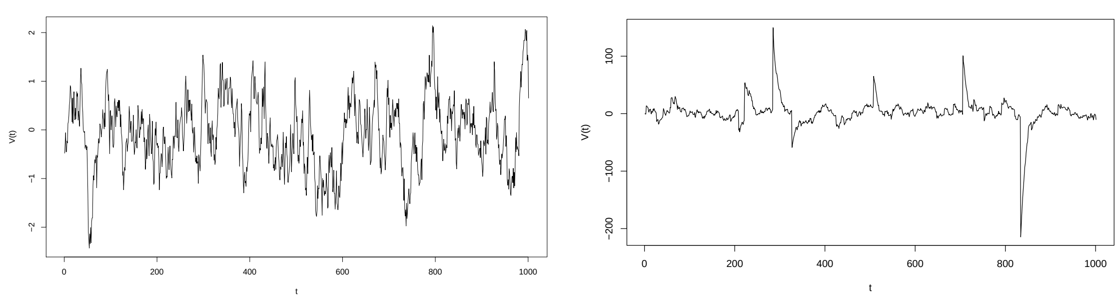
In this section we consider a Monte Carlo simulation study to analyze the behavior of the three different estimation procedures for the γ memory function parameter presented in Section 4.

For the simulation of the solution process $(V(t);\ t \geq 0)$ given by expression (2.1) in the interval $[0, T]$, we consider the discrete version $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$ and the following discrete recursive formula

$$V_{th} = e^{-\gamma h} V_{(t-1)h} + \sigma\, e^{-\gamma h} \sum_{i=1}^{m} e^{\gamma s_i} (B_{s_i} - B_{s_{i-1}}), \tag{5.1}$$

for m sufficiently large, where the sequence of real numbers $(s_i)_{i=1}^{m}$ is a equally spaced partition of the interval $[0, h]$, for any $h \in [0, T]$ and $\{B_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$ is a discrete version of the Brownian motion noise process.

Figures 3(a) and (b) show two time series, of size $n = 1000$, derived from the discrete recursive formula (5.1), when $s_i = \frac{1}{i}$ is the equally spaced partition of the interval $[0, h]$. Figure 3(a) shows a Brownian motion noise process while Figure 3(b) shows the case of a α-stable Lévy noise process with $\alpha = 1.5$. In both graphs the memory function parameter is $\gamma = 0.9$.



(a) (b)

Figure 3: Time Series, of size $n = 1000$, derived from (5.1), with $\gamma = 0.9$, when the noise process is a: (a) Brownian motion; (b) α-stable Lévy process with $\alpha = 1.5$.

5.1 Estimation Results for γ

For the estimation procedures we consider two estimators proposed by [25] given, respectively, by expressions (4.3) and (4.4), and the ordinary least squares estimator given in expression (4.5).

The estimation results were obtained based on time series $\{V_t\}_{t=1}^{n}$, of sample size n, derived from the expression (5.1), with small and large sample sizes, i.e., $n \in \{1000, 10000\}$, to analyze the small sample properties of these three estimators. In both sample sizes the estimators are the average of $re \in \{100, 500\}$ replications. These replication values are good enough since the process $(V(t);\ t \in [0, T])$ is ergodic [28]. We also have results when $n = 20000$; however, they did not show a significant improvement for the parameter γ estimation.

From Tables 1 and 2 one can observe that both estimators $\hat{\gamma}_1$ and $\hat{\gamma}_3$ have better performance, in the sense of small mean squared error, when compared to $\hat{\gamma}_2$. The estimator $\hat{\gamma}_2$ has always more bias than the other two, except when $\gamma = 0.1$. Nevertheless, all three estimators improve when the sample size increases.

Table 1: Estimation Results for the Parameter γ, when $\sigma = 1$, $m = 1000$ and $L \sim S_{1.0}(1, 0, 0)$, for $n \in \{1000, 10000\}$.

γ	$\hat{\gamma}_1$	bias($\hat{\gamma}_1$)	mse($\hat{\gamma}_1$)	$\hat{\gamma}_2$	bias($\hat{\gamma}_2$)	mse($\hat{\gamma}_2$)	$\hat{\gamma}_3$	bias($\hat{\gamma}_3$)	mse($\hat{\gamma}_3$)
$n = 1000$, $h = 1$, $m^* = 1000$, $re = 100$									
0.1	0.1019	0.0019	0.0001	0.0956	-0.0044	0.0003	0.0995	-0.0005	0.0001
0.4	0.4046	0.0046	0.0004	0.3019	-0.0981	0.0100	0.4021	0.0021	0.0003
0.9	0.9087	0.0087	0.0030	0.5594	-0.3406	0.1171	0.9017	0.0017	0.0018
$n = 1000$, $h = 1$, $m^* = 1000$, $re = 500$									
0.1	0.1040	0.0040	0.0001	0.0984	-0.0016	0.0003	0.1014	0.0014	0.0001
0.4	0.4039	0.0039	0.0006	0.3020	-0.0980	0.0103	0.4005	0.0005	0.0005
0.9	0.9111	0.0111	0.0056	0.5650	-0.3350	0.1148	0.9074	0.0074	0.0055
$n = 10000$, $h = 0.1$, $m^* = 2000$, $re = 100$									
0.1	0.1023	0.0023	0.0002	0.1057	0.0057	0.0005	0.0997	-0.0003	0.0002
0.4	0.4042	0.0042	0.0004	0.3875	-0.0125	0.0012	0.4009	0.0009	0.0003
0.9	0.9112	0.0112	0.0016	0.8454	-0.0546	0.0088	0.9086	0.0086	0.0016
$n = 10000$, $h = 0.1$, $m^* = 2000$, $re = 500$									
0.1	0.1038	0.0038	0.0002	0.1091	0.0091	0.0006	0.1012	0.0012	0.0002
0.4	0.4102	0.0102	0.0143	0.3966	-0.0034	0.0032	0.4011	0.0011	0.0004
0.9	0.9020	0.0020	0.0007	0.8338	-0.0662	0.0068	0.8997	-0.0003	0.0007

5.2 Estimation Results for α

Let us consider the equation (2.1) with memory function as a delta Dirac function, as in example 2. From equation (2.6) we have

$$e^{\gamma h} V_{th} - V_{(t-1)h} = \sigma \int_0^h e^{\gamma s} dL(s). \tag{5.2}$$

Table 2: Estimation Results for the Parameter γ, when $\sigma = 1$, $m = 1000$ and $L \sim S_{1.5}(1, 0, 0)$, for $n \in \{1000, 10000\}$.

γ	$\hat{\gamma}_1$	bias($\hat{\gamma}_1$)	mse($\hat{\gamma}_1$)	$\hat{\gamma}_2$	bias($\hat{\gamma}_2$)	mse($\hat{\gamma}_2$)	$\hat{\gamma}_3$	bias($\hat{\gamma}_3$)	mse($\hat{\gamma}_3$)
$n = 1000$, $h = 1$, $m^* = 1000$, $re = 100$									
0.1	0.1057	0.0057	0.0002	0.1006	0.0006	0.0005	0.1020	0.0020	0.0001
0.4	0.4120	0.0120	0.0017	0.3094	-0.0905	0.0090	0.4054	0.0054	0.0008
0.9	0.9013	0.0013	0.0023	0.5557	-0.3442	0.1200	0.8983	-0.0016	0.0023
$n = 1000$, $h = 1$, $m^* = 1000$, $re = 500$									
0.1	0.1050	0.0050	0.0002	0.0993	-0.0006	0.0004	0.1015	0.0015	0.0001
0.4	0.4063	0.0063	0.0009	0.3025	-0.0974	0.0104	0.4031	0.0031	0.0008
0.9	0.9074	0.0074	0.0036	0.5615	-0.3384	0.1161	0.9037	0.0037	0.0034
$n = 10000$, $h = 0.1$, $m^* = 2000$, $re = 100$									
0.1	0.1021	0.0021	0.0001	0.1039	0.0039	0.0004	0.0994	-0.0005	0.0001
0.4	0.4032	0.0032	0.0006	0.3988	-0.0011	0.0019	0.4003	0.0003	0.0006
0.9	0.9098	0.0098	0.0016	0.8373	-0.0626	0.0067	0.9072	0.0072	0.0016
$n = 10000$, $h = 0.1$, $m^* = 2000$, $re = 500$									
0.1	0.1043	0.0043	0.0001	0.1088	0.0088	0.0005	0.1013	0.0013	0.0001
0.4	0.4050	0.0050	0.0007	0.3942	-0.0057	0.0018	0.4017	0.0017	0.0006
0.9	0.9045	0.0045	0.0016	0.8354	-0.0645	0.0086	0.9016	0.0016	0.0015

For the integral in (5.2) one can use the following approximation

$$\sigma \int_0^h e^{\gamma s} dL(s) \approx \sigma \sum_{i=1}^{m} e^{\gamma s_i} (L_{s_i} - L_{s_{i-1}}), \tag{5.3}$$

for m sufficiently large, where the sequence of real numbers $(s_i)_{i=1}^m$ is a equally spaced partition of the interval $[0, h]$, for any $h \in [0, T]$ and $(L(t);\ t \in [0, T])$ is a Lévy process.

To estimate the parameter α we consider the two-step algorithm given in Section 4.2, based on the empirical characteristic function of the process $\{V_t\}_{t=0}^{\lfloor \frac{T}{h} \rfloor}$. As in, [29] we choose $t_1 = 0.2$ and $t_2 = 0.8$ in expression (4.6).

Table 3 presents the estimation results for the parameter α when a Lévy flight process is the noise in the solution to GLE, for $\alpha \in \{1.1, 1.2, \cdots, 1.9\}$, $n \in \{1000, 10000\}$ and $re = 100$. From this table one can observe that when $n = 1000$, the best estimate for α happens when $\gamma = 0.1$, in the sense of small mean squared error values. However, when $n = 10000$, the estimation for α improved considerably. The estimates for α are always very good whatever is the value of γ.

6 Conclusions

In this work we have shown how to use the concept of stochastic integral in the sense of convergence in probability to obtain a stochastic process V which represents the solution to

Table 3: Estimation Results for the α Parameter for Different Values of γ, when $\sigma = 1$, $m = 1000$ and $L \sim S_\alpha(1, 0, 0)$.

α	γ	$\hat{\gamma}$	$\hat{\alpha}$	bias($\hat{\alpha}$)	mse($\hat{\alpha}$)
$n = 1000$, $h = 1$					
1.1	0.1	0.1039	1.1011	0.0011	0.0043
1.1	0.4	0.4077	1.0846	-0.0154	0.0050
1.1	0.9	0.9047	1.0519	-0.0481	0.0148
1.5	0.1	0.1042	1.4995	-0.0005	0.0037
1.5	0.4	0.4080	1.4947	-0.0053	0.0043
1.5	0.9	0.9011	1.4508	-0.0492	0.0120
1.9	0.1	0.1039	1.9039	0.0039	0.0014
1.9	0.4	0.4042	1.9014	0.0014	0.0031
1.9	0.9	0.9084	1.7716	-0.1284	0.0298
$n = 10000$, $h = 0.1$					
1.1	0.1	0.1032	1.0966	-0.0034	0.0005
1.1	0.4	0.4033	1.1049	0.0049	0.0004
1.1	0.9	0.9009	1.1011	0.0011	0.0003
1.5	0.1	0.1014	1.4978	-0.0022	0.0003
1.5	0.4	0.4025	1.4998	-0.0002	0.0003
1.5	0.9	0.9094	1.4993	-0.0007	0.0003
1.9	0.1	0.1008	1.8991	-0.0009	0.0002
1.9	0.4	0.4019	1.9010	0.0010	0.0002
1.9	0.9	0.9011	1.8980	-0.0020	0.0001

the Generalized Langevin Equation. Our proposal was motivated by previous representation formulas obtained using methods such as Laplace transform, Itô's stochastic calculus and ideas developed by Kannan in [13]. All these works are restricted to second order processes. Our approach is of particular interest for modeling phenomena when the noise can have infinite second moment [1, 5, 6, 9, 3, 2, 7, 8, 4].

Using these concepts, probabilistic properties of the solution process V were obtained. In order to numerically verify the convergence in probability, we also developed a numerical method to perform the stochastic integration. The explicit formula for the characteristic function derived in the Appendix was used to numerically study the PDF of V. The numerical results have shown excellent agreement.

In order to apply statistical techniques from time series analysis, we have presented three estimations methods for parameters related to the process V in a particular example. Monte Carlo simulation result showing the estimation of the memory function parameter was performed.

Appendix A

Let the integrator L appearing in (2.1) be a Lévy process. Therefore, the stochastic process $L = (L(t),\ t \geq 0)$ satisfies:

Condition 1: $L(0) = 0$ with probability 1;

Condition 2: L has independent and stationary increments, i.e. $L(t_1), L(t_2) - L(t_1), \cdots, L(t_n) - L(t_{n-1})$ are independent random variables for $0 < t_1 < t_2 < \cdots < t_n$ and $L(t+h) - L(t)$ has the same probability distribution as $L(h)$;

Condition 3: L is continuous in probability, that is, given $t \geq 0$ and $\varepsilon > 0$, we have

$$\lim_{h \to 0} \mathbb{P}\left(|L(t+h) - L(t)| > \varepsilon\right) = 0. \tag{A.1}$$

Under such conditions, the random variable $L(1)$ has an infinitely divisible distribution and its characteristic function is given by (2.3) with $t = 1$ [15, 30].

To construct the *stochastic integral in the sense of convergence in probability* used in (2.1) consider an arbitrary interval $[a, b]$ and for each integer $n \geq 1$, denote

$$\begin{aligned}\Pi_n(\xi) = \{a = t_0 < \cdots < t_n = b, \xi = (\xi_1, \cdots, \xi_n);& \\ \xi_k \in [t_{k-1}, t_k], k = 1, \cdots, n\}&\end{aligned}$$

where $a = t_0 < t_1 < \cdots < t_n = b$ is a subdivision of $[a, b]$ and ξ_k is a chosen point from $[t_{k-1}, t_k]$. Let $\|\Pi_n(\xi)\| = \max\{t_k - t_{k-1}, k = 1, \cdots, n\}$ and suppose that $\lim\limits_{n \to +\infty} \|\Pi_n(\xi)\| = 0$.

Define the sequence of Stieltjes sums as

$$S_n\left(\Pi_n(\xi)\right) = \sum_{k=1}^{n} \rho(\xi_k)\left(L(t_k) - L(t_{k-1})\right). \tag{A.2}$$

If the sequence $\{S_n\left(\Pi_n(\xi)\right),\ n \geq 1\}$ converges in probability to a random variable S and independently of the choice of the partition $\Pi_n(\xi)$ we say that the stochastic integral of ρ with respect to L exists in the sense of convergence in probability and denote it by

$$S = \int_a^b \rho(t) dL(t). \tag{A.3}$$

Theorem A.1 provides conditions to ensure the existence of such integrals.

Theorem A.1[30, pp. 148] *Let $L = (L(t),\ t \geq 0)$ be a stochastic process satisfying conditions 1, 2 and 3 before. If ρ is a continuous real-valued function on $[a, b]$, then the stochastic integral in (A.3) exists in the sense of convergence in probability.*

Assuming the existence of (A.3) its characteristic function is given in Proposition A.1.

Proposition A.1: *Let be L be a stochastic process satisfying the conditions of Theorem A.1. If ρ is continuous on $[a,b]$ and ψ is the characteristic exponent of $L(1)$, then the characteristic function of S in A.3 is given by*

$$\varphi(\lambda) = \exp\left\{-\int_a^b \psi(\lambda\rho(s))ds\right\}. \tag{A.4}$$

Proof:

For each integer $n \geq 1$ consider the partition $\Pi_n(\xi)$ of $[a,b]$. Let $\varphi_n(\lambda)$ be the characteristic function of the Stieltjes sum S_n (A.2). Then, using the independence of the increments $\Delta L_k = L(t_k) - L(t_{k-1})$ and (2.3), we have

$$\begin{aligned}\varphi_n(\lambda) = & \prod_{k=1}^{n} \mathbb{E}\left[\exp\left\{i\lambda\rho(\xi_k)\Delta L_k\right\}\right] = \\ = & \exp\left\{-\sum_{k=1}^{n} \psi\left(\lambda\rho(\xi_k)\right)\Delta t_k\right\},\end{aligned}$$

where $\mathbb{E}[.]$ stands for the expectation (average) operator. Since the sequence $\{S_n\}$ converges in probability to the stochastic integral S we also have convergence in distribution, that is, the sequence of characteristic functions $\varphi_n(\lambda)$ converges to the characteristic function $\varphi(\lambda)$ of the stochastic integral S.

Using the continuity of ψ and ρ, we have the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \psi\left(\lambda\rho(\xi_k)\right)\Delta t_k = \int_a^b \psi\left(\lambda\rho(t)\right)dt.$$

So, for every real λ,

$$\lim_{n\to\infty} \varphi_n(\lambda) = \exp\left\{-\int_a^b \psi\left(\lambda\rho(t)\right)dt\right\},$$

Then, (A.4) follows by the Lévy Continuity Theorem [15]. For details and other related results see [20]. □

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