Continuous Processes Derived from the Solution of Generalized Langevin Equation: Theoretical Properties and Estimation

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Abstract

In this paper we present a class of continuous-time processes arising from the solution of the generalized Langevin equation and show some of its properties. We define the theoretical and empirical codifference as a measure of dependence for stochastic processes. As an alternative dependence measure we also consider the spectral covariance. These dependence measures replace the autocovariance function when it is not well defined. Results for the theoretical codifference and theoretical spectral covariance functions for the mentioned process are presented. The maximum likelihood estimation procedure is proposed to estimate the parameters of the process arising from the classical Langevin equation, i.e., the Ornstein-Uhlenbeck process, and of the so-called Cosine process. We also present a simulation study for particular processes arising from this class showing the generation, and the theoretical and empirical counterpart for both codifference and spectral covariance measures.

Keywords: Generalized Langevin Equation; Codifference and Spectral Covariance; Stable Processes; Maximum Likelihood Estimation Method.

1 Introduction

The classical Langevin equation defines a continuous stochastic process. It was introduced by Langevin (1908) to model the motion dynamics of a particle immersed in a fluid medium. It is given by

$$\begin{cases} dV(t) = -\theta V(t)dt + dL(t) \\ V(0) = V_0, \end{cases} \tag{1.1}$$

where $\theta > 0$ is a constant of friction and $\{L(t)\}_{t\geq 0}$ is a noise process, representing a random force.

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This equation can be solved by applying the Laplace transform methods (or Ito's formula). From this perspective, the solution of (1.1) is given by

$$V(t) = V_0 e^{-\theta t} + \int_0^t e^{-\theta(t-s)}\, dL(s). \tag{1.2}$$

The stochastic process solution $\{V(t)\}_{t\geq 0}$, given in (1.2), is called the Ornstein-Uhlenbeck (OU) process. It is widely used for modeling financial time series, such as interest and exchange rates, as well as other applications. For more details regarding the OU process, see Barndorff-Nielsen and Shephard (2003), Barndorff-Nielsen and Shephard (2001), Barndorff-Nielsen and Shephard (2000), Jongbloed et. al. (2005), and Zhang and Zhang (2013).

In 1965 Hazime Mori proposed a generalization of the classical Langevin equation. Another generalization was proposed by Ryogo Kubo in 1966, which became known as the generalized Langevin equation (GLE). This equation is given by

$$\begin{cases} dV(t) = -\int_0^t \gamma(t-s)\, V(s)\, ds\, dt + d\, L(t) \\ V(0) = V_0, \end{cases} \tag{1.3}$$

where $\{L(t)\}_{t\geq 0}$ is a noise process, V_0 is a random variable independent of $L(\cdot)$ and $\gamma(\cdot)$ is the memory function. Assuming that all processes are second order moments, that is, they have finite quadratic mean, Kannan (1977) studied the solution of GLE. The author showed that any mean square solution $\{V(t)\}_{t\geq 0}$ process of the GLE has the form

$$V(t) = V_0 \rho(t) + \int_0^t \rho(t-s)\, dL(s),$$

where V_0 is a random variable such that $V_0 = V(0)$, $\{L(t)\}_{t\geq 0}$ is the noise process and $\rho(\cdot)$ is a deterministic function satisfying the Volterra integro-differential equation, given by

$$\begin{cases} \rho'(t) = -\int_0^t \gamma(t-s)\rho(s)\, ds \\ \rho(0) = 1. \end{cases} \tag{1.4}$$

The subject of this paper is to study a continuous process derived from the GLE solution, considering the Lévy process as the noise process. In order to generalize the solution class of the GLE, we should modify the function $\rho(\cdot)$. Instead of (1.4), we consider another integro-differential equation that is given in Definition 3.1. This idea extends the previous work done by Medino et al. (2012). Another goal is to investigate the dependence structure of the process, since the autocovariance function is not well defined in the case of infinite second moment processes. We propose to use the so-called codifference as a dependence measure and analyze the properties of its estimator. The codifference function was introduced by Astrauskas (1983) and has been studied by many authors. We also investigate an alternative dependence measure, the so-called spectral covariance, introduced by Paulauskas (1976) and we consider its estimator. Besides, we are interested in studying the parameter estimation of these processes. In this work we present the estimation procedure based on the maximum likelihood for two processes: the OU process and the so-called Cosine process.

The paper is organized as follows: Section 2 presents the codifference and spectral covariance dependence measures and their estimators. Section 3 presents the generalization of expression (1.4) and the class of processes obtained from this equation. Section 4 presents examples of this class. In special, a recurrence formula is derived for particular cases of the general process. Simulated time series are generated using this recurrence formula and we present their theoretical and empirical counterpart for both codifference and spectral covariance measures. Section 5 presents a Monte Carlo simulation study for the maximum likelihood estimation of the process arising from the classical Langevin equation (Ornstein-Uhlenbeck process) and of the so-called Cosine process. Section 6 concludes the paper.

2 Dependence Measures: Codifference and Spectral Covariance

In this section we present two dependence measures: the *codifference* and the *spectral covariance*. The theoretical codifference and its empirical counterpart are defined in Subsection 2.1. We prove the estimator consistency for stationary symmetric α-stable processes that satisfy a mild condition. The spectral covariance is defined in Subsection 2.2 together with an estimator for it based on the spectral measure estimation.

2.1 Codifference Function

In this subsection we want to define a dependence measure for any process. Let X_1 and X_2 be two random variables. The codifference of X_1 and X_2 is defined as

$$\begin{aligned}\tau(X_1, X_2) = &\ln\{\mathbb{E}[\exp(i(X_1 - X_2))]\} - \ln\{\mathbb{E}[\exp(i(X_1))]\} \\ &- \ln\{\mathbb{E}[\exp(-i(X_2))]\}. \end{aligned} \tag{2.1}$$

The codifference function, defined in (2.1), is related to the function considered by Astrauskas (1983). This measure was used again in Astrauskas et al. (1991).

Remark 2.1. (a) If X_1 and X_2 are independent random variables, then $\tau(X_1, X_2) = 0$.
(b) If X_1 and X_2 are Gaussian random variables, then $\tau(X_1, X_2) = \text{Cov}(X_1, X_2)$.
(c) The codifference function is well defined even when the process does not have finite mean. The codifference function given in (2.1) was proposed by Kokoszka and Taqqu (1995).

If $\{X(t)\}_{t\geq 0}$ is any process, then the codifference function is given by

$$\tau_X(k, t) = \tau(X(k), X(t)), \tag{2.2}$$

for $k, t \geq 0$. For more details, we refer the reader to Samorodnitsky and Taqqu (1994).

There is an even more general definition for the codifference, similar to the one proposed in Kokoszka and Taqqu (1994), given by

$$\begin{aligned}\tau_X(s; k, t) = &\ln\{\mathbb{E}[\exp(is(X(t+k) - X(t)))]\} - \ln\{\mathbb{E}[\exp(is(X(t+k)))]\} \\ &- \ln\{\mathbb{E}[\exp(-is(X(t)))]\}, \end{aligned} \tag{2.3}$$

where $s \in \mathbb{R}$, $k \geq 0$ and $t \geq 0$. When $s = 1$, expression (2.3) reduces to (2.1).

Remark 2.2. If $\{X(t)\}_{t\geq 0}$ is any stationary process, then expression (2.3) does not depend on t. In this situation, expression (2.3) will be denoted by $\tau_X(s; k)$.

We will consider the codifference function estimator proposed in Rosadi and Deistler (2009), for ARMA processes. In this work we want to consider any stationary process with symmetric α-stable finite-dimensional distributions and prove the estimator consistency. Let $\{X(t)\}_{t\geq 0}$ be any stationary process and let $\{X_i\}_{i=1}^{N}$ be a sample of size N derived from this process. As the codifference function is defined via characteristic functions, it can be estimated by empirical characteristic functions. The estimator for the codifference function at k, proposed by Rosadi and Deistler (2009), is given by

$$\hat{\tau}_X(s;k) = \sqrt{\frac{N-k}{N}}\left[\ln\left(\frac{1}{N-k}\sum_{t=1}^{N-k} e^{is(X_{t+k}-X_t)}\right) - \ln\left(\frac{1}{N-k}\sum_{t=1}^{N-k} e^{isX_{t+k}}\right) - \ln\left(\frac{1}{N-k}\sum_{t=1}^{N-k} e^{-isX_t}\right)\right], \tag{2.4}$$

for any $k \in \{0, \cdots, N\}$. For more details on the estimator given in (2.4), we refer the reader to Rosadi and Deistler (2009).

The consistency property of the empirical codifference is given in Theorem 2.1. We need to consider the following condition to derive this consistency property:

Condition A: $\tau_X(s;k) \to 0$, when $k \to \infty$, for all $s \in \mathbb{R}$.

Notice that Condition A is not so strong, since at least stationary stable processes that present the mixing property must satisfy this condition (see Gross, 1994).

Let us defined the k-th difference of the $\{X(t)\}_{t\geq 0}$ process by

$$W(t) = X(t+k) - X(t). \tag{2.5}$$

Theorem 2.1. *Let $\{X(t)\}_{t\geq 0}$ be any stationary symmetric α-stable process, $0 < \alpha \leq 2$, satisfying Condition A. Let $\{W(t)\}_{t\geq 0}$ be the process defined in (2.5) and assume it also satisfies Condition A, for any fixed k. For $s \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, the sample codifference $\hat{\tau}_X(s;k)$, defined in expression (2.4), is a consistent estimator for the theoretical codifference $\tau_X(s;k)$, when $N \to \infty$.*

To show the consistency property of the codifference estimator, first it is necessary to prove the following two lemmas.

Lemma 2.1. *Let $\{X(t)\}_{t\geq 0}$ be any stationary symmetric α-stable process, $0 < \alpha \leq 2$, satisfying Condition A, and denote by $\Phi_X(s) = \mathbb{E}(e^{isX(t)})$ its characteristic function. For $s \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$,*

$$\ln(\hat{\phi}(s;k)) := \ln\left(\frac{1}{N-k}\sum_{t=1}^{N-k} e^{isX_t}\right)$$

is a consistent estimator for $\ln(\Phi_X(s))$, when $N \to \infty$.

Proof: Let $Y_s(t) := e^{isX(t)}$. Notice that the process $\{Y_s(t)\}_{t\geq 0}$ is stationary. For simplicity, instead of working with $\hat{\phi}(s;k)$, we first show the consistency property for $\hat{\phi}^*(s) := \frac{1}{N}\sum_{t=1}^{N} e^{isX_t}$.

Here, $\hat{\phi}^*(s)$ is an unbiased estimator for $\Phi_X(s) = \mathbb{E}(Y_s(t))$. To show the weak consistency for this estimator, we show that $Y_s(t)$ is a mean ergodic process. A sufficient condition for $Y_s(t)$ to be a mean ergodic process, i.e. $\hat{\phi}^*(s) \to \Phi_X(s)$ in the mean square sense, is that its covariance function tends to zero when k tends to infinity (see theorem 7.1.1 in Brockwell and Davis, 1987). The covariance function of $Y_s(t)$ at k can be expressed as

$$\begin{aligned} C_{Y_s}(k) &= \mathrm{Cov}(Y_s(t+k), Y_s(t)) = \mathbb{E}(Y_s(t+k)\overline{Y_s(t)}) - \mathbb{E}(Y_s(t+k))\mathbb{E}(\overline{Y_s(t)}) \\ &= \mathbb{E}(e^{isX(t+k)}e^{-isX(t)}) - |\Phi_X(s)|^2 = |\Phi_X(s)|^2 \left(\frac{\mathbb{E}(e^{is(X(t+k)-X(t))})}{\mathbb{E}(e^{isX(t+k)})\mathbb{E}(e^{-isX(t)})} - 1 \right). \end{aligned} \tag{2.6}$$

Notice that

$$\exp(\tau_X(s;k)) = \frac{\mathbb{E}(e^{is(X(t+k)-X(t))})}{\mathbb{E}(e^{isX(t+k)})\mathbb{E}(e^{-isX(t)})}.$$

Then, we have

$$C_{Y_s}(k) = |\Phi_X(s)|^2 \left(\exp(\tau_X(s;k)) - 1\right).$$

From Condition A, $C_{Y_s}(k) \to 0$, when $k \to \infty$. As mean square convergence entails convergence in probability, we have $\hat{\phi}^*(s) \xrightarrow{\mathbb{P}} \Phi_X(s)$, for all $s \in \mathbb{R}$. Moreover, $\Phi_X(\cdot)$ is a positive real-valued function, since we are considering symmetric α-stable processes. Therefore, we can conclude that $\mathrm{Re}\,(\hat{\phi}^*(s)) \xrightarrow{\mathbb{P}} \mathrm{Re}\,(\Phi_X(s)) = \Phi_X(s)$ and $\mathrm{Im}\,(\hat{\phi}^*(s)) \xrightarrow{\mathbb{P}} \mathrm{Im}\,(\Phi_X(s)) = 0$, where $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ are the real and imaginary parts of $z \in \mathbb{C}$.

By taking the principal value of the $\ln(\cdot)$ function in the complex domain, this function will be continuous and well-defined on $\mathbb{C}$ minus the negative real line. It is possible to see that $\mathrm{Re}\,(\Phi_X(s)) = \Phi_X(s) > 0$, but $\mathrm{Re}\,(\hat{\phi}^*(s))$ can be less than or equal to zero. Therefore, without loss of generality, we restrict the definition of the real and imaginary parts of $\ln(\hat{\phi}^*(s))$ only on the right half plane where $\mathrm{Re}\,(\hat{\phi}^*(s))$ is greater than zero, and equal to zero, otherwise. From this consideration, given that $\ln(z) = \ln|z| + i\arg(z)$, it is possible to obtain

$$\begin{aligned} \mathrm{Re}\,[\ln(\hat{\phi}^*(s))] &= \ln\{[\mathrm{Re}\,(\hat{\phi}^*(s))]^2 + [\mathrm{Im}\,(\hat{\phi}^*(s))]^2\}^{\frac{1}{2}} \\ &= \frac{1}{2}\ln\{[\mathrm{Re}\,(\hat{\phi}^*(s))]^2 + [\mathrm{Im}\,(\hat{\phi}^*(s))]^2\} \end{aligned}$$

and

$$\mathrm{Im}\,[\ln(\hat{\phi}^*(s))] = \arctan\left[\frac{\mathrm{Im}\,(\hat{\phi}^*(s))}{\mathrm{Re}\,(\hat{\phi}^*(s))}\right].$$

From the continuity of the logarithm function in the considered domain, we have $\mathrm{Re}\,[\ln(\hat{\phi}^*(s))] \xrightarrow{\mathbb{P}} \mathrm{Re}\,[\ln(\Phi_X(s))] = \ln(\Phi_X(s))$ and $\mathrm{Im}\,[\ln(\hat{\phi}^*(s))] = \arg(\hat{\phi}^*(s)) \xrightarrow{\mathbb{P}} 0$, when $N \to \infty$. In other words, $\ln(\hat{\phi}^*(s)) \xrightarrow{\mathbb{P}} \ln(\Phi_X(s))$. To complete the proof, it is sufficient to show that $\hat{\phi}^*(s) - \hat{\phi}(s;k) \xrightarrow{\mathbb{P}} 0$. We can see that

$$\begin{aligned} \mathbb{E}|\hat{\phi}^*(s) - \hat{\phi}(s;k)| &= \mathbb{E}\left|\frac{1}{N}\sum_{t=1}^{N} e^{isX_t} - \frac{1}{N-k}\sum_{t=1}^{N-k} e^{isX_t}\right| \\ &= \mathbb{E}\left|\left(\frac{1}{N} - \frac{1}{N-k}\right)\sum_{t=1}^{N-k} e^{isX_t} + \frac{1}{N}\sum_{t=N-k+1}^{N} e^{isX_t}\right| \\ &\leq \left|\frac{1}{N} - \frac{1}{N-k}\right|\sum_{t=1}^{N-k} |e^{isX_t}| + \frac{1}{N}\sum_{t=N-k+1}^{N} |e^{isX_t}| = \left(\frac{1}{N-k} - \frac{1}{N}\right)(N-k) + \frac{k}{N} = \frac{2k}{N}. \end{aligned}$$

Thus, when $N \to \infty$, $\hat{\phi}^*(s) - \hat{\phi}(s;k) \to 0$ in mean, and it also converges in probability. □

Remark 2.3. If X is an α-stable random variable denoted by $X \sim S_\alpha(\sigma, \beta, \mu)$, then $Y = e^{isX}$ is not an α-stable random variable, for every fixed $s \in \mathbb{R}$. Indeed, the variance of Y is finite. Thus, the expression (2.6) is well defined.

Corollary 2.1. *Let $\{X(t)\}_{t\geq 0}$ be any stationary symmetric α-stable process, $0 < \alpha \leq 2$. Let $Y_s(t) = e^{isX(t)}$. Suppose that the autocovariance function of the process $\{Y_s(t)\}_{t\geq 0}$, denoted by $C_{Y_s}(\cdot)$, is such that $C_{Y_s}(k) \to 0$, when $k \to \infty$. Then, the codifference function $\tau_X(s;k)$, defined in expression* (2.3)*, is asymptotically zero, when $k \to \infty$.*

Proof: From the expression (2.6), we have

$$C_{Y_s}(k) = |\Phi_X(s)|^2 \left(\frac{\mathbb{E}(e^{is(X(t+k)-X(t))})}{\mathbb{E}(e^{isX(t+k)})\mathbb{E}(e^{-isX(t)})} - 1 \right) \to 0,$$

when $k \to \infty$. Then,

$$\begin{aligned} &\frac{\mathbb{E}(e^{is(X(t+k)-X(t))})}{\mathbb{E}(e^{isX(t+k)})\mathbb{E}(e^{-isX(t)})} - 1 \to 0 \Longleftrightarrow \frac{\mathbb{E}(e^{is(X(t+k)-X(t))})}{\mathbb{E}(e^{isX(t+k)})\mathbb{E}(e^{-isX(t)})} \to 1 \\ &\Longleftrightarrow \ln \left(\frac{\mathbb{E}(e^{is(X(t+k)-X(t))})}{\mathbb{E}(e^{isX(t+k)})\mathbb{E}(e^{-isX(t)})} \right) \to 0. \end{aligned} \tag{2.7}$$

Notice that the left-hand side term of (2.7) is $\tau_X(s;k)$. Hence, $\tau_X(s;k) \to 0$, when $k \to \infty$.

□

Lemma 2.2. *Let $\{X(t)\}_{t\geq 0}$ be any stationary symmetric α-stable process satisfying Condition A. Let $\{W(t)\}_{t\geq 0}$ be the process defined in* (2.5) *and assume it also satisfies Condition A, for any fixed k. Let $\Phi_W(s;k) = \mathbb{E}(e^{is(X(t+k)-X(t))})$ denote the characteristic function of $\{W(t)\}_{t\geq 0}$. For $k \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{R}$*

$$\ln(\hat{\phi}(s;k)) \xrightarrow{\mathbb{P}} \ln(\Phi_W(s;k)),$$

when $N \to \infty$, where $\hat{\phi}(s;k)$ is given by

$$\hat{\phi}(s;k) := \frac{1}{N-k} \sum_{t=1}^{N-k} e^{is(X_{t+k}-X_t)}.$$

Proof: For the proof, we can proceed in a similar way as in Lemma 2.1. Firstly we show the consistency property for $\hat{\phi}^*(s;k) := \frac{1}{N} \sum_{t=1}^{N} e^{is(X_{t+k}-X_t)}$. Define $Z(t) := Y_s(t+k)\overline{Y_s(t)} = e^{is(X(t+k)-X(t))}$, for fixed k. It is easy to see that $\{Z(t)\}_{t\geq 0}$ is a stationary process, that is $Z(t+l) \stackrel{d}{=} Z(t)$. We shall show that $Z(\cdot)$ is a mean ergodic process. A sufficient condition for $Z(\cdot)$ to be a mean ergodic process is that its covariance function tends to zero. The covariance function of $Z(\cdot)$ at lag l can be given as

$$\begin{aligned} C_Z(l) &= \mathrm{Cov}(Z(t+l), Z(t)) = \mathbb{E}(Z(t+l)\overline{Z(t)}) - \mathbb{E}(Z(t+l))\mathbb{E}(\overline{Z(t)}) \\ &= \mathbb{E}(e^{is(X(t+l+k)-X(t+l)-X(t+k)+X(t))}) - |\Phi_W(s;k)|^2 \\ &= \mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))}) - |\Phi_W(s;k)|^2. \end{aligned} \tag{2.8}$$

Then, we need to show that $\mathbb{E}(Z(t+l)\overline{Z(t)}) = \mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))}) \to |\Phi_W(s;k)|^2$, when $l \to \infty$. Notice that

$$\begin{aligned}
\mathbb{E}(Z(t+l)\overline{Z(t)}) &= |\Phi_W(s;k)|^2 \left(\frac{\mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))})}{|\Phi_W(s;k)|^2} \right) \\
&= |\Phi_W(s;k)|^2 \left(\frac{\mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))})}{\mathbb{E}(e^{is(X(t+l+k)-X(t+l))})\mathbb{E}(e^{-is(X(t+k)-X(t))})} \right) \\
&= |\Phi_W(s;k)|^2 \exp(\tau_W(s;l)), \qquad (2.9)
\end{aligned}$$

where $\tau_W(s;l) = \ln\left(\frac{\mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))})}{\mathbb{E}(e^{is(X(t+l+k)-X(t+l))})\mathbb{E}(e^{-is(X(t+k)-X(t))})} \right)$ is the codifference function of $W(\cdot)$, for fixed k and t. By hypothesis, we have $\tau_W(s;l) \to 0$ when $l \to \infty$. Hence $\exp(\tau_W(s;l)) \to 1$. In other words, $\mathbb{E}(e^{is((X(t)-X(t+k))-(X_{t+l}-X_{t+l+k})}) \to |\Phi_W(s;k)|^2$ when $l \to \infty$, and therefore $\hat{\phi}^*(s;k)$ converges in mean square to $\Phi_W(s;k)$. Therefore, $\hat{\phi}^*(s;k) \xrightarrow{\mathbb{P}} \Phi_W(s;k)$. For the remaining of this proof, we can proceed similarly to the proof of Lemma 2.1. □

Corollary 2.2. *Let $\{X(t)\}_{t\geq 0}$ be any stationary symmetric α-stable process, $0 < \alpha \leq 2$. Let $Z(t) := e^{is(X(t+k)-X(t))}$, for fixed k. Suppose that the autocovariance function of the process $\{Z(t)\}_{t\geq 0}$, denoted by $C_Z(\cdot)$, is such that $C_Z(l) \to 0$, when $l \to \infty$. Let $\{W(t)\}_{t\geq 0}$ be the process defined in* (2.5)*. Then, the codifference function $\tau_W(s;l)$, defined in expression* (2.3)*, is asymptotically zero when $l \to \infty$.*

Proof: From the expression (2.8), we have

$$C_Z(l) = \mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))}) - |\Phi_W(s;k)|^2 \to 0,$$

when $l \to \infty$. Hence,

$$\mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))}) \to |\Phi_W(s;k)|^2.$$

From the expression (2.9),

$$\begin{aligned}
&|\Phi_W(s;k)|^2 \frac{\mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))})}{\mathbb{E}(e^{is(X(t)-X(t+k))})\mathbb{E}(e^{is(X(t+l+k)-X(t+l))})} \to |\Phi_W(s;k)|^2 \\
&\iff \ln\left[\frac{\mathbb{E}(e^{is((X(t)-X(t+k))-(X(t+l)-X(t+l+k))})}{\mathbb{E}(e^{is(X(t)-X(t+k))})\mathbb{E}(e^{is(X(t+l+k)-X(t+l))})} \right] \to 0. \qquad (2.10)
\end{aligned}$$

Notice that the left-hand side term in (2.10) is $\tau_W(s;l)$, where $W(\cdot)$ is defined by (2.5), for fixed k. Therefore, $\tau_W(s;l) \to 0$ when $l \to \infty$.

□

Proof of Theorem 2.1: For fixed k and $N \to \infty$, we have $\sqrt{\frac{N-k}{N}} \to 1$. From Lemmas 2.1 and 2.2, it is true that

$$\hat{\tau}_X(s;k) \xrightarrow{\mathbb{P}} \ln[\Phi_W(s;k)] - \ln[\Phi_X(s)] - \ln[\Phi_X(-s)] = \tau_X(s;k),$$

for $s \in \mathbb{R}$, when $N \to \infty$.

□

2.2 Spectral Covariance

As an alternative dependence measure for random variables with infinite variance, we can also consider the spectral covariance. This dependence measure was introduced by Paulauskas (1976) and it was revisited in Damarackas and Paulauskas (2014).

Given an α-stable random vector (X_1, X_2), with $0 < \alpha < 2$, $\alpha \neq 1$, we define the spectral covariance as

$$\varrho(X_1, X_2) = \int_{S_2} s_1\, s_2\, \Gamma(d\mathbf{s}), \tag{2.11}$$

where Γ is the spectral measure on $S_2 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$. The advantage of using the spectral covariance is its definition based only on the spectral measure, not on the characteristic function. Damarackas and Paulauskas (2014) provided an analysis, based on some examples, for the best α parameter dependence of the spectral covariance.

Consider an α-stable stochastic process given in the integral form as follows

$$X(t) = \int_E f_t(s) d\, L(s), \tag{2.12}$$

where $E \subseteq \mathbb{R}$ is a set, $\{L(t)\}_{t \geq 0}$ is the α-stable Lévy process and $\{f_t(\cdot)\}_{t\geq 0}$ is such that $\int_E |f_t(s)|^\alpha ds < \infty$. In this case, Damarackas and Paulauskas (2014) showed that the spectral covariance can be written as

$$\varrho(X(t), X(t+k)) = \int_E f_t(s) f_{t+k}(s) \|\bar{f}(s)\|^{\alpha-2}\, ds, \tag{2.13}$$

where $\|\bar{f}(s)\|^2 = f_t^2(s) + f_{t+k}^2(s)$.

For estimation purposes, we will consider the estimator proposed in Kodia and Garel (2014), where for any fixed $t \geq 0$, we have

$$\hat{\varrho}(X(t), X(t+k)) = \sum_{j=1}^{m} \hat{\sigma}_{j,k} \cos\left(\frac{2\pi(j-1)}{m}\right) \sin\left(\frac{2\pi(j-1)}{m}\right), \tag{2.14}$$

where $\hat{\boldsymbol{\sigma}}_k = (\hat{\sigma}_{j,k})_{j=1}^m$ such that $\hat{\boldsymbol{\sigma}}_k = \min_{\boldsymbol{\sigma}\geq 0} \|\hat{I}_k - \hat{\Psi}\boldsymbol{\sigma}\|$. To estimate the weights $\hat{\boldsymbol{\sigma}}_k$ consider $\hat{\Psi}$ an $m \times m$ matrix defined by $\hat{\Psi} = (\hat{\psi}_\alpha(\langle \mathbf{t_j}, \mathbf{s_l}\rangle))_{j,l=1}^m$ such that $\hat{\psi}_\alpha(\langle \mathbf{t_j}, \mathbf{s_l}\rangle) = |t_{j1}s_{l1} + t_{j2}s_{l2}|^{\hat{\alpha}}$, where $\hat{\alpha}$ is some estimate for α and $\mathbf{t_j} = \mathbf{s_j} = \left(\cos\left(\frac{2\pi(j-1)}{m}\right), \sin\left(\frac{2\pi(j-1)}{m}\right)\right)$. In this work we use four estimators for α: the maximum likelihood (denoted by $\hat{\alpha}_{mle}$), the regression-type estimator proposed by Koutrouvelis (1980) (denoted by $\hat{\alpha}_{kou}$), the quantile based estimator of McCulloch (1986) (denoted by $\hat{\alpha}_{mc}$) and the regression-type estimator proposed by Press (1972) (denoted by $\hat{\alpha}_{pr}$). The estimator defined in expression (2.14) requires an i.i.d. sample $\mathbf{X_k}^{(1)}, \cdots, \mathbf{X_k}^{(re)}$ of $(X(t), X(t+k))$, where re is the number of replications. Let $\hat{I}_k = (\hat{I}_{k,re}(\mathbf{t_j}))_{j=1}^m$ and $\hat{I}_{k,re}(\mathbf{t_j}) = -\ln(\hat{\phi}_{k,re}(\mathbf{t_j}))$, where $\hat{\phi}_{k,re}(\mathbf{t_j})$ is the empirical characteristic function given by $\hat{\phi}_{k,re}(\mathbf{t_j}) = \frac{1}{re}\sum_{j=1}^{re} e^{i\langle \mathbf{t_j}, \mathbf{X_k}^{(j)}\rangle}$. For more details on the empirical spectral covariance, we refer the reader to Kodia and Garel (2014).

3 Processes Derived from the GLE Solution

We introduce in this section our procedure to study the Generalized Langevin Equation or the GLE, for short, and present what we call the *Generalized Langevin Process*. Such way to consider this equation is specially useful in cases where the noise has infinite second moment, but it also can be applied in the finite second moment situations. The main element in our method is Definition 3.1, where we consider GLE driven by Lévy processes. For the background noise process, we assume stochastic processes $L = \{L(t)\}_{t\geq 0}$ satisfying the following conditions:

B1: $L(0) \equiv 0$ with probability 1.
B2: L has independent increments, i.e., $L(t_0), L(t_1) - L(t_0), \cdots, L(t_n) - L(t_{n-1})$ are independent random variables for every $0 < t_0 < t_1 < \cdots < t_{n-1} < t_n$ for all positive integer n.
B3: L has stationary increments, i.e., for all $t \geq 0$, $L(t+h) - L(t)$ has the same distribution as $L(h)$, for all $h > 0$.
B4: L is continuous in probability, that is, given $t \geq 0$ and $\delta > 0$, we have

$$\lim_{h\to 0} \mathbb{P}(|L(t+h) - L(t)| > \delta) = 0.$$

For a treatment on Lévy processes suitable for the scope of this paper, we refer the reader to Applebaum (2009) or Schoutens (2003). We recall that the only Lévy process that has finite second moment is the standard Brownian motion, also known as the Wiener process, and that the random variable $L(1)$ has infinitely divisible distribution whose characteristic function is given by

$$\varphi_L(x) = e^{-\psi(x)},$$

where $\psi : \mathbb{R} \to \mathbb{C}$ is the characteristic exponent of $L(1)$.

Remark 3.1. If $\{L(t)\}_{t\geq 0}$ is the symmetric α-stable Lévy process, then the characteristic exponent of $L(1)$ is given by $\psi(x) = |x|^\alpha$.

Definition 3.1. Let $V = \{V(t)\}_{t\geq 0}$ be a stochastic process and $\rho = \{\rho(t)\}_{t\geq 0}$ be a deterministic function. We say that the pair (V, ρ) represents a solution to the GLE if V is given by

$$V(t) = V_0\rho(t) + \int_0^t \rho(t-s)\, dL(s) \tag{3.1}$$

and the function ρ satisfies the following integro-differential equation

$$\begin{cases} \rho'(t) = -\int_0^t \rho(s)\, d\mu_t(s), \\ \rho(0) = 1, \end{cases} \tag{3.2}$$

where $\{\mu_t\}_{t\geq 0}$ is a family of signed measures and $L = \{L(t)\}_{t\geq 0}$ is a Lévy process. The stochastic process $V = \{V(t)\}_{t\geq 0}$ will be called the Generalized Langevin Process.

Under the conditions in Definition 3.1, the stochastic integral in (3.1) can be taken in the sense of convergence in probability if $\rho(\cdot)$ is continuous (Lukacs, 1975) or, in the general setting, considering stochastic integration with respect to semimartingale (Applebaum, 2009). Integro-differential equations as in (3.2) are well studied in Mingarelli (1983).

The following proposition gives a characterization of the discrete form of the process in (3.1) and will be useful for numerical, simulation and estimation purposes.

Proposition 3.1. *Under the conditions in Definition 3.1 and if* $\{L(t)\}_{t\geq 0}$ *is a standard* α*-stable Lévy motion, the process given by* (3.1) *has the following discrete form*

$$V(n+1) - V(n) \stackrel{d}{=} V_0\left(\rho(n+1) - \rho(n)\right) + \xi_n, \tag{3.3}$$

where $\stackrel{d}{=}$ *means equality in distribution and* $\xi_n \sim S_\alpha(\sigma_n, 0, 0)$*, such that* σ_n *is given by*

$$\sigma_n^\alpha = \int_0^n |\rho(n+1-s) - \rho(n-s)|^\alpha\, ds + \int_n^{n+1} |\rho(n+1-s)|^\alpha\, ds. \tag{3.4}$$

Proof: From the expression (3.1) one has

$$V(n+1) - V(n) = V_0\left(\rho(n+1) - \rho(n)\right) + \xi_n,$$

where $\xi_n = \int_0^{n+1} \rho(n+1-s)\, dL(s) - \int_0^n \rho(n-s)\, dL(s)$. We can rewrite ξ_n as follows

$$\xi_n = \int_0^n [\rho(n+1-s) - \rho(n-s)]\, dL(s) + \int_n^{n+1} \rho(n+1-s)\, dL(s) = A_n + B_n, \tag{3.5}$$

where $A_n = \int_0^n [\rho(n+1-s) - \rho(n-s)]\, dL(s)$ and $B_n = \int_n^{n+1} \rho(n+1-s)\, dL(s)$, such that A_n and B_n are independent. In addition, by Proposition 3.4.1 in Samorodnitsky and Taqqu (1994), $A_n \sim S_\alpha(\sigma_{A_n}, 0, 0)$ and $B_n \sim S_\alpha(\sigma_{B_n}, 0, 0)$, where

$$\sigma_{A_n}^\alpha = \int_0^n |\rho(n+1-s) - \rho(n-s)|^\alpha\, ds, \tag{3.6}$$

$$\sigma_{B_n}^\alpha = \int_n^{n+1} |\rho(n+1-s)|^\alpha\, ds. \tag{3.7}$$

By Property 1.2.1 in Samorodnitsky and Taqqu (1994), we have $\xi_n \sim S_\alpha(\sigma_n, 0, 0)$, where σ_n is given by (3.4).

□

Let $\{I_\rho(t)\}_{t\geq 0}$ be the stochastic process given by $I_\rho(t) = \int_0^t \rho(t-x) dL(x)$ with $\tau_{I_\rho}(s; k, t)$ as its codifference function. The proposition A.1 in Medino et. al. (2012) says that characteristic functions of stochastic integrals are given in terms of the integrand function and the characteristic exponent of $L(1)$. Then, from Remark 3.1, $\tau_{I_\rho}(s; k)$ can be rewriting as

$$\begin{aligned}
\tau_{I_\rho}(s; k) &= \ln\left[\frac{\mathbb{E}\big(e^{is\int_0^t (\rho(t+k-x) - \rho(t-x))dL(x)}\big)\mathbb{E}\big(e^{is\int_t^{t+k} \rho(t+k-x)dL(x)}\big)}{\mathbb{E}\big(e^{is\int_0^{t+k} \rho(t+k-x)dL(x)}\big)\mathbb{E}\big(e^{-is\int_0^t \rho(t-x)dL(x)}\big)}\right] \\
&= \ln\left[\frac{e^{-|s|^\alpha\left(\int_0^t |\rho(t+k-x) - \rho(t-x)|^\alpha dx + \int_t^{t+k} |\rho(t+k-x)|^\alpha dx\right)}}{e^{-|s|^\alpha\left(\int_0^{t+k} |\rho(t+k-x)|^\alpha dx + \int_0^t |\rho(t-x)|^\alpha dx\right)}}\right] \\
&= |s|^\alpha \int_0^t \left(|\rho(t+k-x)|^\alpha + |\rho(t-x)|^\alpha - |\rho(t+k-x) - \rho(t-x)|^\alpha\right) dx.
\end{aligned} \tag{3.8}$$

The next proposition gives the general formula for the codifference function of the stochastic process defined by (3.1).

Proposition 3.2. *Let* $\{V(t)\}_{t\geq 0}$ *be the stochastic process defined in the expression* (3.1)*. Then the following statements hold.*

(i) The codifference function of $\{V(t)\}_{t\geq 0}$ *is given by*

$$\tau_V(s;k,t) = \ln\left[\frac{\varphi_{V_0}(s(\rho(t+k)-\rho(t)))}{\varphi_{V_0}(s\rho(t+k))\varphi_{V_0}(-s\rho(t))}\right] + \tau_{I_\rho}(s;k,t),$$

where $\varphi_{V_0}(\cdot)$ *is the characteristic function of the random variable* $V_0 \equiv V(0)$.

(ii) If $\{V(t)\}_{t\geq 0}$ *is stationary, then its codifference function reduces to*

$$\tau_V(s;k) = \ln\left[\frac{\varphi_{V_0}(s(\rho(k)-1))}{\varphi_{V_0}(s\rho(k))\varphi_{V_0}(-s)}\right],$$

where $\varphi_{V_0}(\cdot)$ *is the characteristic function of the random variable* $V_0 \equiv V(0)$.

(iii) Let $\{L(t)\}_{t\geq 0}$ *be a symmetric* α*-stable Lévy process and* $V_0 \sim S_\alpha(\sigma,0,0)$. *Then the codifference function of* $\{V(t)\}_{t\geq 0}$ *is given by*

$$\begin{aligned}\tau_V(s;k,t) &= |s|^\alpha\sigma^\alpha\left[|\rho(t+k)|^\alpha + |\rho(t)|^\alpha - |\rho(t+k)-\rho(t)|^\alpha\right] \\ &+|s|^\alpha\int_0^t \left(|\rho(t+k-x)|^\alpha + |\rho(t-x)|^\alpha - |\rho(t+k-x)-\rho(t-x)|^\alpha\right)dx. \end{aligned} \tag{3.9}$$

(iv) Let $\{L(t)\}_{t\geq 0}$ *be a symmetric* α*-stable Lévy process and* $V_0 \sim S_\alpha(\sigma,0,0)$. *If* $\{V(t)\}_{t\geq 0}$ *is a stationary process, then*

$$\tau_V(s;k) = |s|^\alpha\sigma^\alpha\left[1+|\rho(k)|^\alpha - |\rho(t+k)-1|^\alpha\right].$$

Proof: *(i)* From (2.3) we have

$$\tau_V(s;k,t) = \ln\left[\frac{\mathbb{E}(e^{is(V(t+k)-V(t))})}{\mathbb{E}(e^{isV(t+k)})\mathbb{E}(e^{-isV(t)})}\right]. \tag{3.10}$$

From expression (3.1) and due the independence between $\{L(t)\}_{t\geq 0}$ and V_0

$$\begin{aligned}\tau_V(s;k,t) &= \ln\left[\frac{\mathbb{E}(e^{isV_0(\rho(t+k)-\rho(t))})\mathbb{E}(e^{is(I_\rho(t+k)-I_\rho(t))})}{\mathbb{E}(e^{isV_0\rho(t+k)})\mathbb{E}(e^{isI_\rho(t+k)})\mathbb{E}(e^{-isV_0\rho(t)})\mathbb{E}(e^{-isI_\rho(t)})}\right] \\ &= \ln\left[\frac{\varphi_{V_0}(s(\rho(t+k)-\rho(t)))}{\varphi_{V_0}(s\rho(t+k))\varphi_{V_0}(-s\rho(t))}\right] + \tau_{I_\rho}(s;k,t), \end{aligned} \tag{3.11}$$

and this completes the proof.

(ii) From the stationarity property, the value t can be taken equal to zero. Then, from the item *(i)*

$$\tau_V(s;k) = \ln\left[\frac{\varphi_{V_0}(s(\rho(k)-1))}{\varphi_{V_0}(s\rho(k))\varphi_{V_0}(-s)}\right] + \tau_{I_\rho}(s;k).$$

Notice that

$$\tau_{I_\rho}(s;k) = \tau(I_\rho(k), I_\rho(0)) = \ln\left(\frac{\mathbb{E}(e^{is\int_0^t \rho(t-x)dL(x)})}{\mathbb{E}(e^{is\int_0^t \rho(t-x)dL(x)})}\right) = 0,$$

and this completes the proof.

(iii) From the characteristic function of $V_0 \sim S_\alpha(\sigma,0,0)$ and from item *(i)*, the expression of $\tau_V(s;k,t)$ is given by

$$\begin{aligned}\tau_V(s;k,t) &= \ln\left[\frac{e^{-\sigma^\alpha|s(\rho(t+k)-\rho(t))|^\alpha}}{e^{-\sigma^\alpha|s\rho(t+k)|^\alpha}e^{-\sigma^\alpha|-s\rho(t)|^\alpha}}\right] + \tau_{I_\rho}(s;k)\\ &= |s|^\alpha\sigma^\alpha\left[|\rho(t+k)|^\alpha + |\rho(t)|^\alpha - |\rho(t+k)-\rho(t)|^\alpha\right] + \tau_{I_\rho}(s;k). \qquad (3.12)\end{aligned}$$

From (3.8) and (3.12), we obtain the expression for the codifference function (3.9).

(iv) From the stationarity property, the value t can be taken equal to zero. Then, from items *(ii)* and *(iii)*

$$\tau_V(s;k) = |s|^\alpha\sigma^\alpha\left[1 + |\rho(k)|^\alpha - |\rho(k)-1|^\alpha\right].$$

□

Observe that the codifference function of any stationary process of the form (3.1) only depends on its characteristic function at time zero and on the memory function $\rho(\cdot)$. The next proposition gives the general formula for the spectral covariance of a stochastic process defined in (3.1).

Proposition 3.3. *Let $\{L(t)\}_{t\geq 0}$ be a symmetric α-stable Lévy process and $V_0 \equiv 0$. Then the spectral covariance of $\{V(t)\}_{t\geq 0}$ defined in expression* (3.1) *is given by*

$$\varrho(V(t),V(t+k)) = \int_0^t \rho(t-s)\rho(t+k-s)\left[\rho^2(t-s)+\rho^2(t+k-s)\right]^{\frac{\alpha-2}{2}} ds. \qquad (3.13)$$

If the process $\{V(t)\}_{t\geq 0}$ is stationary, then the expression (3.13) *will depend only on k and t can be considered a fixed value.*

Proof: By equation (3.1) with $V_0 \equiv 0$, we have

$$V(t) = \int_0^\infty \mathbb{I}_{[0,t]}(s)\rho(t-s)\,dL(s). \qquad (3.14)$$

Then equation (2.13) gives

$$\begin{aligned}\varrho(V(t),V(t+k)) &= \int_0^\infty \mathbb{I}_{[0,t]}(s)\rho(t-s)\mathbb{I}_{[0,t+k]}(s)\rho(t+k-s)\left[\mathbb{I}_{[0,t]}(s)^2\rho^2(t-s)\right.\\ &\qquad\left.+\mathbb{I}_{[0,t+k]}(s)^2\rho^2(t+k-s)\right]^{\frac{\alpha-2}{2}} ds\\ &= \int_0^t \rho(t-s)\rho(t+k-s)\left[\rho^2(t-s)+\rho^2(t+k-s)\right]^{\frac{\alpha-2}{2}} ds. \qquad (3.15)\end{aligned}$$

□

To end this section, we observe the if $\theta > 0$ and $\mu_t(E) = \theta\,\mathbb{I}_E(t)$, where $\mathbb{I}_E(\cdot)$ is the indicator function of the set E, then $\{\mu_t\}_{t\geq 0}$ is a family of Dirac measures each of them assigning mass $\theta > 0$ to the point $t \geq 0$. From the expression (3.2), we have $\rho(t) = e^{-\theta t}$, and the resulting process is the well-known Ornstein-Uhlenbeck process.

Also, notice that if $\mu_t = \mu$ for all $t \geq 0$ in Definition 3.1 and μ is absolutely continuous with respect to the Lebesgue measure λ, that is, $d\,\mu_t(s) = d\,\mu(s) = f(s)\,ds$, for all $t, s \geq 0$, where

$$f(s) = \frac{d\,\mu}{d\,\lambda}(s)$$

is the Radon-Nikodym derivative of μ with respect to λ, then expression (3.2) reduces to

$$\begin{cases} \rho''(t) + \rho(t)\, f(t) = 0 \\ \rho'(0) = 0, \quad \rho(0) = 1. \end{cases} \tag{3.16}$$

In the general situation, $d\,\mu_t(s) = \gamma(t-s)\,ds$ will depend on $t \geq 0$ and we recover expression (1.4). This is not the main focus of this manuscript and we will discuss it in a future work.

We want to emphasize that the processes arising from Definition 3.1 are not necessarily Markov. In fact, only the OU process is Markov. But this feature does not hinder to study problems involving this type of process (see Fleming et al., 2014). In the next sections, for some particular functions $f(\cdot)$, we solve the second order initial value problem given in (3.16) and perform some numerical simulation and estimation studies of the resulting Generalized Langevin Processes.

4 Examples

In this section we present examples of processes defined in (3.1) for different functions $\rho(\cdot)$ satisfying expression (3.16). For all examples we consider that $\{L(t)\}_{t\geq 0}$ is the symmetric α-stable Lévy process, which satisfies Conditions **B1-B4**.

Example 4.1. Ornstein-Uhlenbeck Process

The process given in (1.2) is called the Ornstein-Uhlenbeck (OU), where function $\rho(\cdot)$ is given by $e^{-\theta t}$, $\theta > 0$. For generating and simulating purposes, a discrete form will be given. Assume that the OU process is observed at discrete times $\{t_k = kh; k = 0, 1, 2, \cdots\}$, where h is the discretization step size. We can obtain a discretization form of the process by using the additivity property with respect to the integration interval, that is,

$$V(kh) = e^{-\theta h} V((k-1)h) + Z_{k,h}, \tag{4.1}$$

where $Z_{k,h} = \int_{(k-1)h}^{kh} e^{\theta(s-kh)} dL(s)$ and $V((k-1)h) = e^{-\theta(k-1)h} V_0 + \int_0^{(k-1)h} e^{\theta(s-(k-1)h)} dL(s)$. Furthermore, by using equality of characteristic functions, we can show that

$$Z_{k,h} \stackrel{d}{=} \left(\frac{1 - e^{-\theta\alpha h}}{\theta\alpha} \right)^{1/\alpha} S_k, \tag{4.2}$$

where $\stackrel{d}{=}$ denotes equality in distribution and $\{S_k\}_{k\in\mathbb{N}}$ is an independent identically distributed (iid) sequence of symmetric α-stable random variables with scale parameter σ.

To calculate the codifference function we need the V_0 distribution. If $\{L(t)\}_{t\geq 0}$ is the standard α-stable Lévy motion, which satisfies conditions **B1-B3**, then the distribution of V_0 can be easily obtained. In Applebaum (2009), the expression (1.2) is rewritten as

$$V(t) = \int_{-\infty}^{t} e^{-\theta(t-s)} dL(s), \tag{4.3}$$

where the integral is defined by taking $\{L(t)\}_{t<0}$ to be an independent copy of $\{-L(t)\}_{t\geq 0}$. From expression (4.3) it follows that $V_0 = \int_{-\infty}^{0} e^{\theta s} dL(s)$. Applying proposition 3.4.1 in Samorodnitsky and Taqqu (1994), one has $V_0 \sim S_\alpha(\tilde{\sigma}, 0, 0)$, where $\tilde{\sigma} = \left(\frac{1}{\theta\alpha}\right)^{1/\alpha}$. Notice that, with these parameters for V_0, the OU process is stationary; however, it is not stationary when V_0 is a constant random variable. The theoretical codifference function of the OU process is given by the following expression (see example 4.7.1 in Samorodnitsky and Taqqu, 1994).

$$\tau_V(s;k) = \frac{|s|^\alpha}{\theta\alpha}\left(1 + e^{-\alpha\theta k} - (1 - e^{-\theta k})^\alpha\right). \tag{4.4}$$

Our simulation study for the empirical codifference of the OU process suggests small values for s and hereafter we shall consider $s = 0.01$. Figure 4.1 presents simulated time series and theoretical and empirical codifference functions for the OU process. Notice that when the θ value increases, the theoretical codifference function decreases to zero very fast. Furthermore, when $\theta = 1$ the empirical codifference function better approaches to its theoretical counterpart.

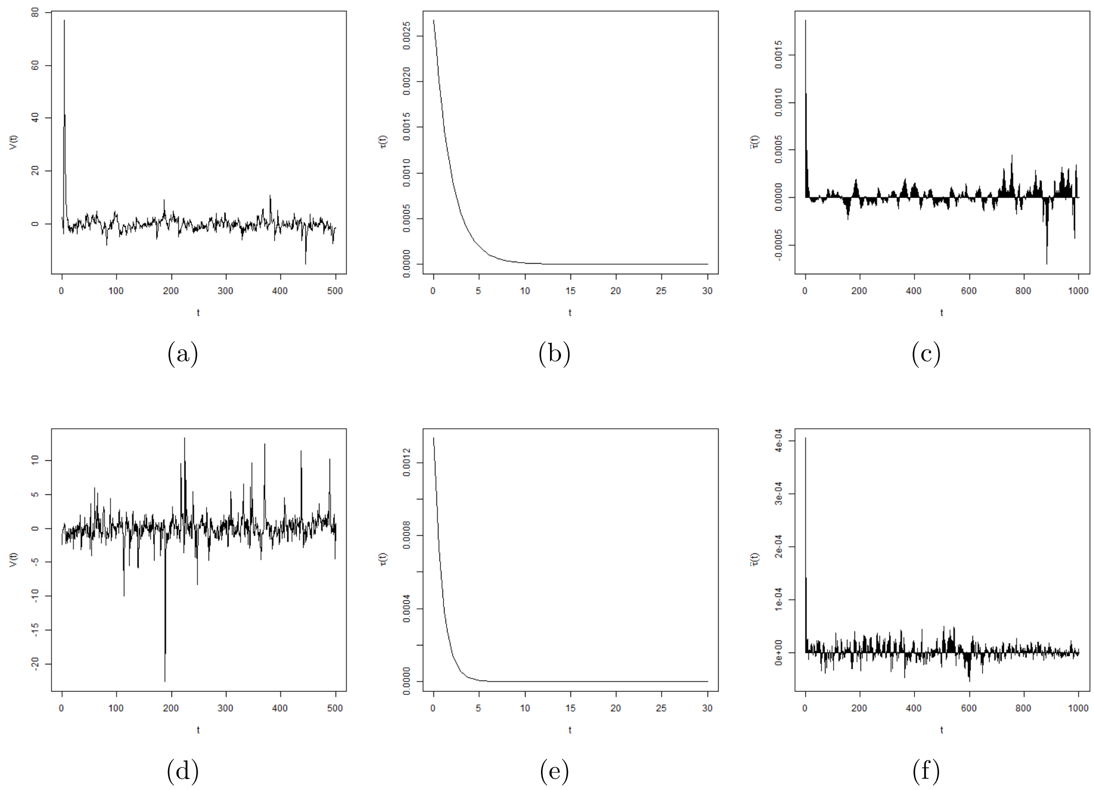


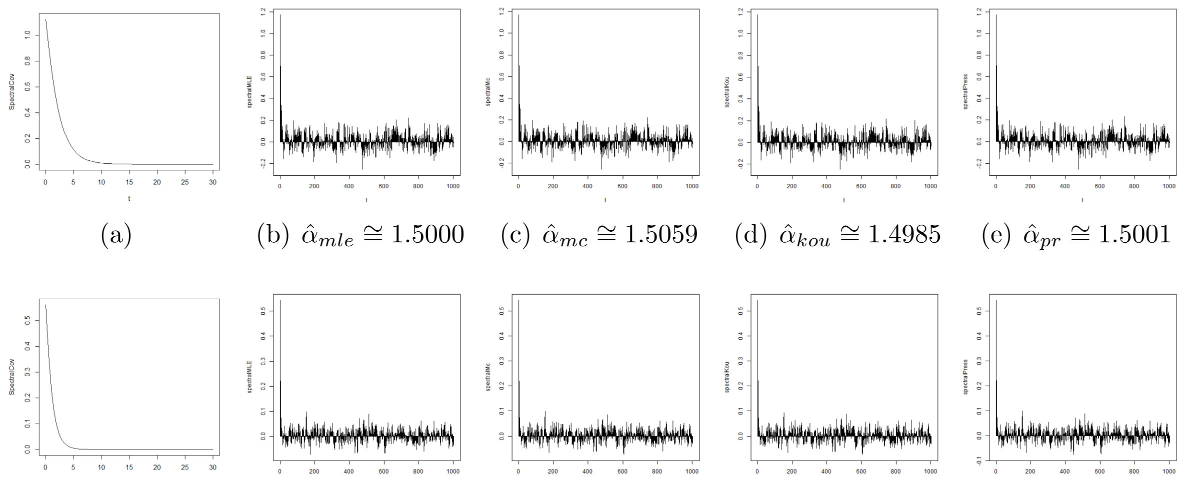
Figure 4.1: Simulated time series, theoretical and empirical codifference functions of the process given in (1.2) when $\alpha = 1.5$, $h = 0.5$, $n = 1000$ and $s = 0.01$. (a), (b) and (c): $\theta = 0.5$ ($\tilde{\sigma} = 1.5396$); (d), (e) and (f): $\theta = 1$ ($\tilde{\sigma} = 0.5443$).

The theoretical spectral covariance of the OU process can be calculated using expression (4.3). It is given by the following expression (see proposition 2 in Damarackas and Paulauskas, 2014)

$$\varrho(V(t), V(t+k)) = \frac{1}{\alpha\theta(1 + e^{-2\theta k})^{(2-\alpha)/2}} e^{-\theta k}, \quad k \geq 0. \tag{4.5}$$

Figure 4.2 presents the theoretical and empirical spectral covariance for the OU process. These graphs present the four different estimators for α, described in Subsection

2.2. We note there is no significant difference in the empirical spectral covariance when the α estimator changes. This is due to the fact that all α estimates are very accurate.



(a) (b) $\hat{\alpha}_{mle} \cong 1.5000$ (c) $\hat{\alpha}_{mc} \cong 1.5059$ (d) $\hat{\alpha}_{kou} \cong 1.4985$ (e) $\hat{\alpha}_{pr} \cong 1.5001$

(f) (g) $\hat{\alpha}_{mle} \cong 1.5001$ (h) $\hat{\alpha}_{mc} \cong 1.5043$ (i) $\hat{\alpha}_{kou} \cong 1.5040$ (j) $\hat{\alpha}_{pr} \cong 1.5066$

Figure 4.2: Theoretical (panels (a) and (f)) and empirical spectral covariance of the OU process given in (1.2) when $\alpha = 1.5$, $\sigma = 1$, $h = 0.5$, $n = 1000$, $re = 1000$ and $t = 0$. (a)-(e): $\theta = 0.5$; (f)-(j): $\theta = 1$.

Example 4.2. Cosine Process

Consider $f(t) = a^2$, for $a > 0$. By solving the differential equation in (3.16) we find $\rho(t) = \cos(at)$ and the resulting process is given by

$$V(t) = V_0 \cos(at) + \int_0^t \cos(a(t-s))\, dL(s). \tag{4.6}$$

Hereafter, we shall call it as the Cosine Process.

The discrete form of this process is given in Proposition 4.1 below.

Proposition 4.1. *Consider the process given in* (4.6)*. One discretization form of this process is given by*

$$V((k+1)h) = 2\,\cos(ah)\, V(kh) - V((k-1)h) + \varepsilon_{k,h}, \tag{4.7}$$

where h is the discretization step size and $\varepsilon_{k,h}$ is $S_\alpha(\sigma_\varepsilon, 0, 0)$ random variable, where

$$\sigma_\varepsilon^\alpha = 2 \int_0^h |\cos(as)|^\alpha ds. \tag{4.8}$$

Proof: From the expression (4.6), we have

$$\begin{aligned} V((k+1)h) =& V_0 \cos(a(k+1)h) + \int_0^{(k+1)h} \cos[a((k+1)h - s)]\, dL(s) \\ =& V_0[\cos(akh)\cos(ah) - \sin(akh)\sin(ah)] \\ &+ \int_0^{(k+1)h} \cos(a(kh-s))\cos(ah) - \sin(a(kh-s))\sin(ah)\, dL(s) \end{aligned}$$

$$= \cos(ah)V(kh) - V_0 \sin(akh)\sin(ah) - \sin(ah)\int_0^{(k+1)h} \sin(a(kh-s))\, dL(s)$$
$$+ \cos(ah)\int_{kh}^{(k+1)h} \cos(a(kh-s))\, dL(s).$$

By using trigonometric properties based on the cosine function and multiplying the resulting expression by two, we obtain

$$\begin{aligned} V((k+1)h) &= 2\cos(ah)V(kh) - \cos(a(k-1)h)V_0 - \int_0^{(k-1)h} \cos(a((k-1)h-s))\, dL(s) \\ &\quad - \int_{(k-1)h}^{(k+1)h} \cos(a((k-1)h-s))\, dL(s) + 2\cos(ah)\int_{kh}^{(k+1)h} \cos(a(kh-s))\, dL(s) \\ &= 2\cos(ah)V(kh) - V((k-1)h) + \varepsilon_{k,h}, \end{aligned}$$

where $\varepsilon_{k,h} = -\int_{(k-1)h}^{(k+1)h} \cos(a((k-1)h-s))\, dL(s) + 2\cos(ah)\int_{kh}^{(k+1)h} \cos(a(kh-s))\, dL(s)$. We can rewrite $\varepsilon_{k,h}$ as

$$\begin{aligned} \varepsilon_{k,h} &= -\int_{(k-1)h}^{kh} \cos(a((k-1)h-s))\, dL(s) \\ &\quad + \int_{kh}^{(k+1)h} [2\cos(ah)\cos(a(kh-s)) - \cos(a((k-1)h-s))]\, dL(s) = A + B, \end{aligned}$$

where $A = -\int_{(k-1)h}^{kh} \cos(a((k-1)h-s))\, dL(s)$ and $B = \int_{kh}^{(k+1)h}[2\cos(ah)\cos(a(kh-s)) - \cos(a((k-1)h-s))]\, dL(s)$. Notice that A and B are independent random variables such that $A \sim S_\alpha(\sigma_A, 0, 0)$, with $\sigma_A^\alpha = \int_0^h |\cos(as)|^\alpha\, ds$, and $B \sim S_\alpha(\sigma_B, 0, 0)$, with $\sigma_B^\alpha = \int_0^h |2\cos(ah)\cos(as) - \cos(a(s+h))|^\alpha\, ds$. Using trigonometric properties, we have $\sigma_B^\alpha = \int_0^h |\cos(a(s-h))|^\alpha\, ds$. Then, by property 1.2.1 in Samorodnitsky and Taqqu (1994), we obtain $\varepsilon_{k,h} \sim S_\alpha(\sigma_\varepsilon, 0, 0)$, with

$$\sigma_\varepsilon^\alpha = \int_0^h |\cos(as)|^\alpha ds + \int_0^h |\cos(a(s-h))|^\alpha ds = 2\int_0^h |\cos(as)|^\alpha ds, \tag{4.9}$$

where the above second equality can be obtained by changing variables.

□

To calculate the codifference function we consider that V_0 is a random variable with symmetric α-stable distribution, denoted by $S_\alpha(\sigma, 0, 0)$. The Cosine process is non-stationary, because there is at least one unit root in the discrete form given in Proposition 4.1. Thus, we can use Proposition 3.2*(iii)* to calculate its theoretical codifference, which is given in Corollary 4.1 below.

Corollary 4.1. *Let* $\{V(t)\}_{t\geq 0}$ *be the process given in* (4.6). *Then its theoretical codifference function is given by*

$$\begin{aligned} \tau_V(s;k,t) =& |s|^\alpha \sigma^\alpha \left[|\cos(a(t+k))|^\alpha + |\cos(at)|^\alpha - |\cos(a(t+k)) - \cos(at)|^\alpha\right] \\ &+ |s|^\alpha \int_0^t (|\cos(a(t+k-x))|^\alpha + |\cos(a(t-x))|^\alpha \\ &- |\cos(a(t+k-x)) - \cos(a(t-x))|^\alpha)dx. \end{aligned} \tag{4.10}$$

Proof: From Proposition 3.2*(iii)*, since $\rho(t) = \cos(at)$, we obtain expression (4.10).

□

Our simulation study for the Cosine process also suggests small values for s and hereafter we shall consider $s = 0.01$. Figure 4.3 presents the simulated time series and theoretical and empirical codifference functions of the process given in (4.6). Notice that when the value of a increases, the theoretical codifference function presents large variability, but preserves the same characteristic. Furthermore, the empirical codifference function does not converge to zero. This was expected, since its theoretical counterpart does not converge to zero either.

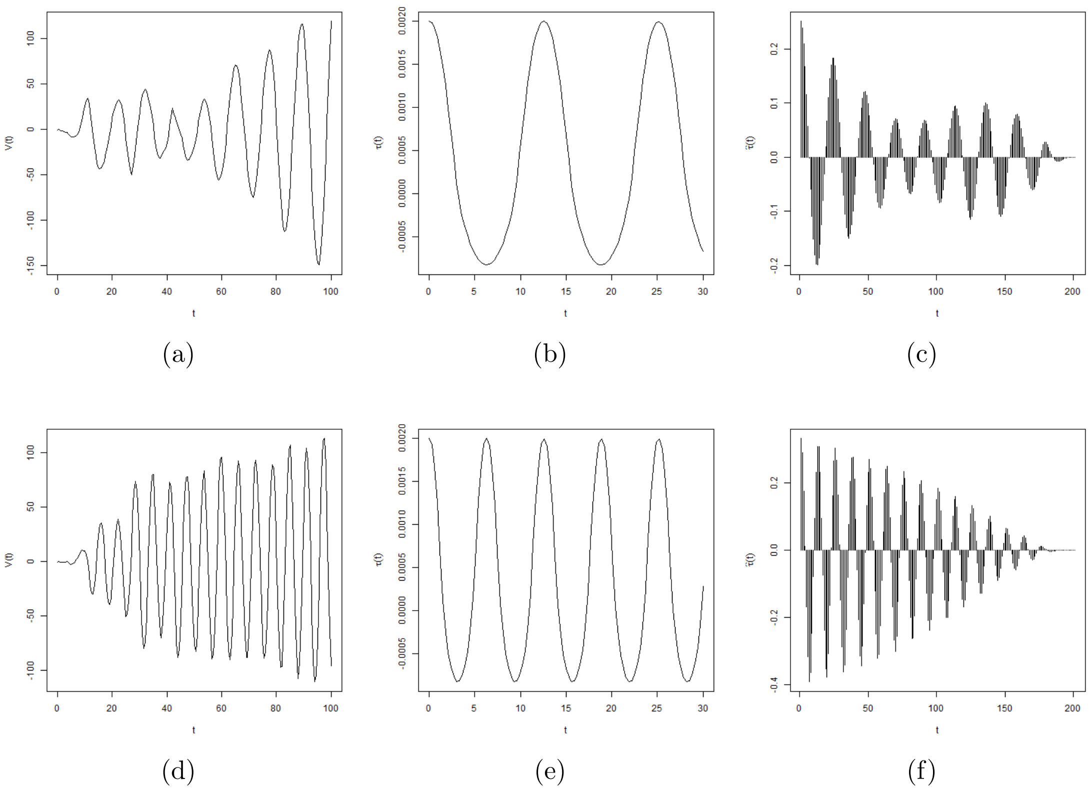


Figure 4.3: Simulated time series, theoretical and empirical codifference functions of the process given in (4.6) when $\alpha = 1.5$, $\sigma = 1$, $h = 0.5$, $n = 200$, $s = 0.01$ and $t = 0$. (a), (b) and (c): $a = 0.5$; (d), (e) and (f): $a = 1$.

The spectral covariance of the Cosine process is given in Corollary 4.2 below.

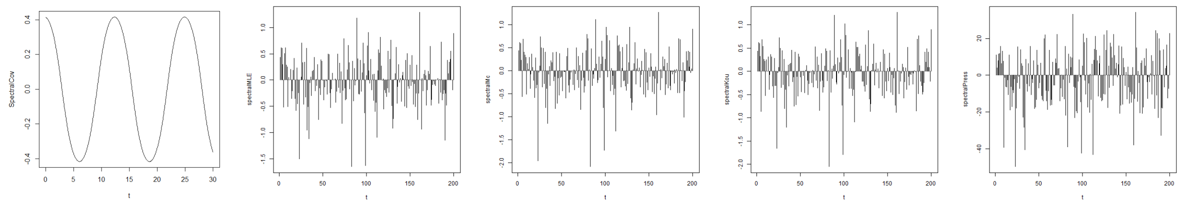
Corollary 4.2. *Let* $\{V(t)\}_{t\geq 0}$ *be the process given in* (4.6)*. Then its spectral covariance is given by*

$$\varrho(V(t), V(t+k)) = \int_0^t \cos(a(t-s))\cos(a(t+k-s))\left[\cos^2(a(t-s)) + \cos^2(a(t+k-s))\right]^{\frac{\alpha-2}{2}} ds. \tag{4.11}$$

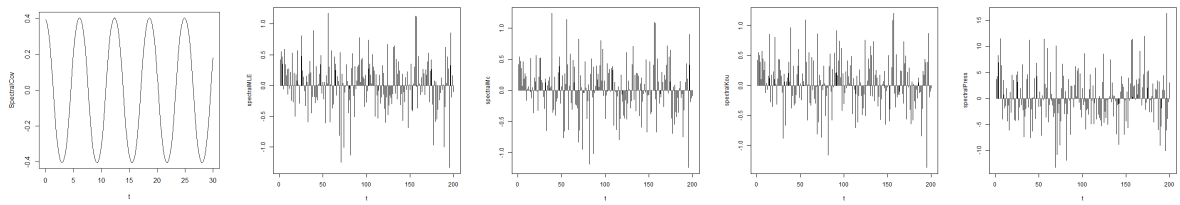
Proof: From Proposition 3.3, since $\rho(t) = \cos(at)$, we obtain expression (4.11).

□

Figure 4.4 presents the theoretical and empirical spectral covariance for the Cosine process. These graphs present the four different estimators for α, described in Subsection 2.2. We note there are differences in the empirical spectral covariance when the α estimator changes. This is due to the fact that each α estimator has different values, especially the one proposed by Press (1972). In this example, $\hat{\alpha}_{pr}$ has the biggest bias.



(a) (b) $\hat{\alpha}_{mle} \cong 1.4985$ (c) $\hat{\alpha}_{mc} \cong 1.5059$ (d) $\hat{\alpha}_{kou} \cong 1.5056$ (e) $\hat{\alpha}_{pr} \cong 0.0262$



(f) (g) $\hat{\alpha}_{mle} \cong 1.4890$ (h) $\hat{\alpha}_{mc} \cong 1.4820$ (i) $\hat{\alpha}_{kou} \cong 1.4971$ (j) $\hat{\alpha}_{pr} \cong 0.0786$

Figure 4.4: Theoretical (panels (a) and (f)) and empirical spectral covariance of the Cosine process given in (4.6) when $\alpha = 1.5$, $\sigma = 1$, $h = 0.5$, $n = 200$, $re = 1000$ and $t = h$. (a)-(e): $a = 0.5$; (f)-(j): $a = 1$.

Example 4.3. Consider $f(t) = 2a(1 - 2at^2)$, for any $a > 0$. By solving the differential equation in (3.16) we find $\rho(t) = e^{-at^2}$ and the resulting process is given by

$$V(t) = V_0 e^{-at^2} + \int_0^t e^{-a(t-s)^2}\, dL(s). \tag{4.12}$$

The discrete form of this process is given in Proposition 4.2 below.

Proposition 4.2. *Consider the process given in* (4.12)*. One discretization form for this process is given by*

$$V((k+1)h) = e^{-a(2k+1)h^2}\, V(kh) + W_{k,h}, \tag{4.13}$$

where h is the discretization step size and

$$W_{k,h} = \int_0^{kh} e^{-a((kh-s)^2+(2k+1)h^2)}(e^{2ash} - 1)\, dL(s) + \int_{kh}^{(k+1)h} e^{-a((kh-s)^2-2sh+(2k+1)h^2)}\, dL(s).$$

Moreover, the distribution of $W_{k,h}$ is $S_\alpha(\sigma_W, 0, 0)$ random variable, where

$$\sigma_W^\alpha = \int_0^{kh} e^{-\alpha a((kh-s)^2+(2k+1)h^2)}(e^{2ash} - 1)^\alpha\, ds + \int_{kh}^{(k+1)h} e^{-\alpha a((kh-s)^2-2sh+(2k+1)h^2)}\, ds. \tag{4.14}$$

Proof: From expression (4.12), we have

$$V_0 e^{-a(kh)^2} = V(kh) - \int_0^{kh} e^{-a(kh-s)^2}\, dL(s). \tag{4.15}$$

We also have

$$V((k+1)h) = e^{-a(2k+1)h^2}\left[V_0\, e^{-a(kh)^2} + \int_0^{(k+1)h} e^{-a((kh)^2-2s(k+1)h+s^2)}\, dL(s)\right].$$

Thus,

$$V_0 e^{-a(kh)^2} = e^{a(2k+1)h^2} V((k+1)h) - \int_0^{(k+1)h} e^{-a((kh)^2-2s(k+1)h+s^2)}\, dL(s). \tag{4.16}$$

From expressions (4.15) and (4.16), we get

$$\begin{aligned} V((k+1)h) =& e^{-a(2k+1)h^2} V(kh) - \int_0^{kh} e^{-a((kh-s)^2+(2k+1)h^2)}\, dL(s) \\ &+ \int_0^{(k+1)h} e^{-a((kh)^2-2s(k+1)h+s^2+(2k+1)h^2)}\, dL(s) \\ =& e^{-a(2k+1)h^2} V(kh) + W_{k,h}, \end{aligned}$$

where

$$\begin{aligned} W_{k,h} =& \int_0^{(k+1)h} e^{-a((kh-s)^2-2sh+(2k+1)h^2)}\, dL(s) - \int_0^{kh} e^{-a((kh-s)^2+(2k+1)h^2)}\, dL(s) \\ =& \int_0^{kh} e^{-a((kh-s)^2+(2k+1)h^2)}(e^{2ash}-1)\, dL(s) + \int_{kh}^{(k+1)h} e^{-a((kh-s)^2-2sh+(2k+1)h^2)}\, dL(s). \end{aligned}$$

Using propositions 1.2.1 and 3.4.1 in Samorodnitsky and Taqqu (1994), the distribution of $W_{k,h}$ is $S_\alpha(\sigma_W, 0, 0)$, where σ_W^α is given by expression (4.14).

□

To calculate the codifference function we consider that V_0 is a random variable with symmetric α-stable distribution, denoted by $S_\alpha(\sigma, 0, 0)$. The theoretical codifference function of this process is given in Corollary 4.3.

Corollary 4.3. *Let* $\{V(t)\}_{t\geq 0}$ *be the process given in* (4.12). *Then its theoretical codifference function is given by*

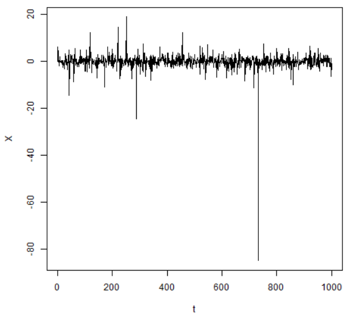
$$\begin{aligned} \tau_V(s;k,t) =& |s|^\alpha \sigma^\alpha \left[e^{-a\alpha(t+k)^2} + e^{-a\alpha t^2} - (e^{-at^2} - e^{-a(t+k)^2})^\alpha\right] \\ &+ |s|^\alpha \int_0^t (e^{-a\alpha(t+k-x)^2} + e^{-a\alpha(t-x)^2} - (e^{-a(t-x)^2} - e^{-a(t+k-x)^2})^\alpha) dx. \end{aligned} \tag{4.17}$$

Proof: From Proposition 3.2*(iii)*, since $\rho(t) = e^{-at^2}$, we obtain expression (4.17).

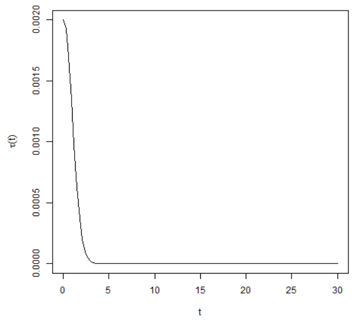
□

The estimation of the theoretical codifference function improves when $s = 0.01$. Figure 4.5 presents simulated time series and theoretical and empirical codifference functions for the process given in (4.12). Notice that, when the value a increases, the theoretical and empirical functions converge quickly to zero. The results are very similar to Example 4.1.

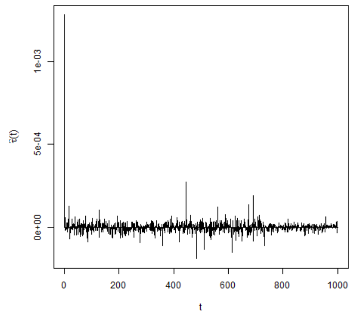
The spectral covariance of the process defined in (4.12) is given in Corollary 4.4 below.



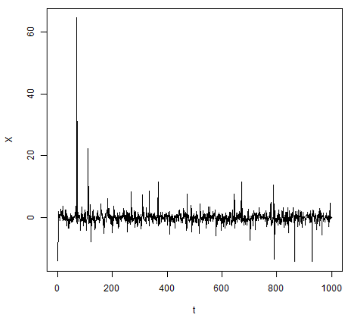
(a)



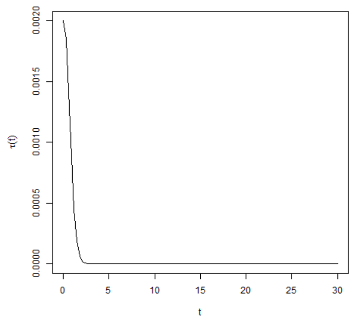
(b) $s = 0.01$



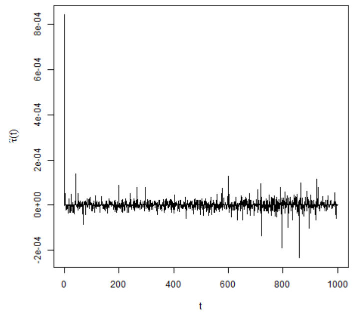
(c) $s = 0.01$



(d)



(e) $s = 0.01$



(f) $s = 0.01$

Figure 4.5: Simulated time series, theoretical and empirical codifference functions of the process given in (4.12) when $\alpha = 1.5$, $\sigma = 1$, $h = 0.5$, $n = 1000$, $s = 0.01$ and $t = 0$. (a), (b) and (c): $a = 0.5$; (d), (e) and (f): $a = 1$.

Corollary 4.4. *Let* $\{V(t)\}_{t\geq 0}$ *be the process defined in* (4.12)*. Then its spectral covariance is given by*

$$\varrho(V(t), V(t+k)) = \int_0^t e^{-a(t-s)^2} e^{-a(t+k-s)^2} \left(e^{-2a(t-s)^2} + e^{-2a(t+k-s)^2}\right)^{\frac{\alpha-2}{2}} ds. \qquad (4.18)$$

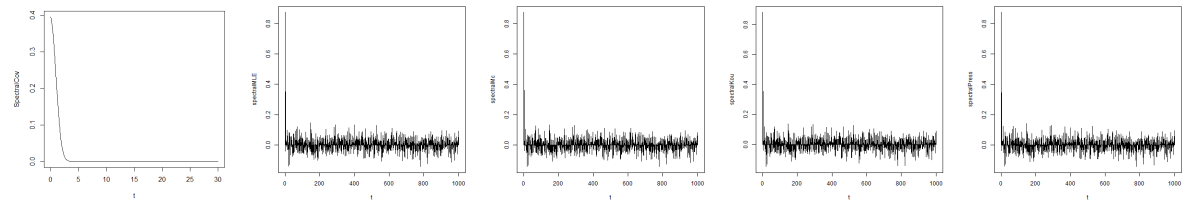
Proof: From Proposition 3.3, since $\rho(t) = e^{-at^2}$, we obtain expression (4.18).

□

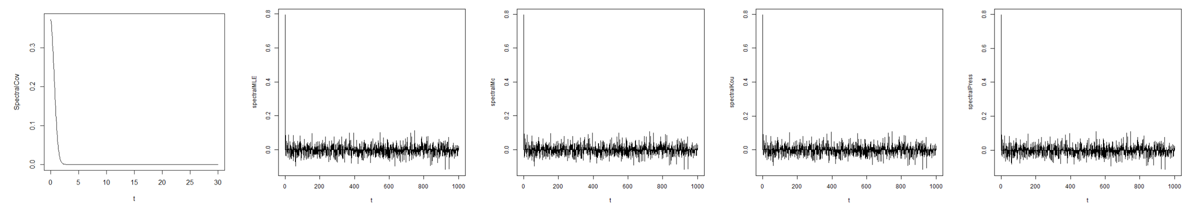
Figure 4.6 presents the theoretical and empirical spectral covariance for the process given in (4.12). These graphs present the four different estimators for α, described in Subsection 2.2. We note there is no significant difference in the empirical spectral covariance when the α estimator changes. This is due to the fact that all α estimates are very accurate.

5 Monte Carlo Simulations

In this section we present Monte Carlo simulation results for the maximum likelihood estimation (mle) in both the OU and Cosine processes.



(a) (b) $\hat{\alpha}_{mle} \cong 1.4974$ (c) $\hat{\alpha}_{mc} \cong 1.5026$ (d) $\hat{\alpha}_{kou} \cong 1.5001$ (e) $\hat{\alpha}_{pr} \cong 1.5015$



(f) (g) $\hat{\alpha}_{mle} \cong 1.4980$ (h) $\hat{\alpha}_{mc} \cong 1.5064$ (i) $\hat{\alpha}_{kou} \cong 1.5046$ (j) $\hat{\alpha}_{pr} \cong 1.5050$

Figure 4.6: Theoretical (panels (a) and (f)) and empirical spectral covariance of the process given in (4.12) when $\alpha = 1.5$, $\sigma = 1$, $h = 0.5$, $n = 1000$, $re = 1000$ and $t = h$. (a)-(e): $a = 0.5$; (f)-(j): $a = 1$.

5.1 Maximum Likelihood in the OU Process

For every function $\rho(\cdot)$ and each noise process $L(\cdot)$, the process given in (3.1) has different parameters. In this subsection, we estimate the parameters for the case when $\rho(t) = e^{-\theta\, t}$, that is, the OU process.

From expression (4.1), notice that we can consider the OU as an AR(1) process. Let $\boldsymbol{\eta} = (\alpha, \sigma, \theta)'$ be the parameter vector to be estimated and let $\{V_{kh}\}_{k=0}^{N-1}$ be a sample of size N of the process given by (1.2). We have

$$Z_{k,h} = V_{kh} - e^{-\theta h} V_{(k-1)h}.$$

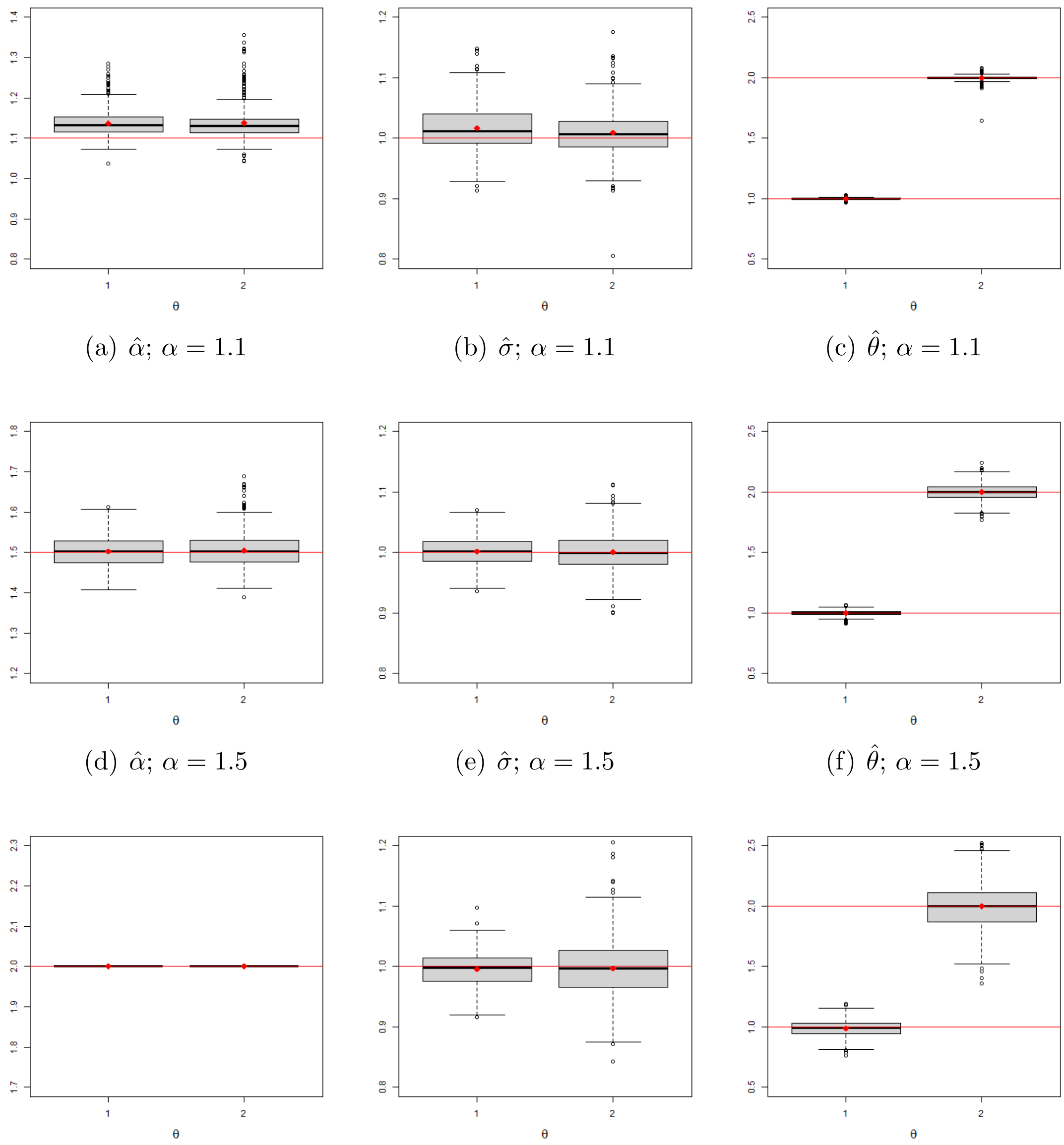
Notice that, for a fixed h, $\{Z_{k,h}\}_{k \in \mathbb{N}}$ is a sequence of i.i.d. random variables with symmetric α-stable distribution and scale parameter σ. Hence, the likelihood function is given by

$$\mathcal{L}(\boldsymbol{\eta} | Z_{1,h}, \cdots, Z_{N-1,h}) = \prod_{k=1}^{N-1} f(Z_{k,h} | \boldsymbol{\eta}),$$

where $f(\cdot|\boldsymbol{\eta})$ is the density distribution function of the α-stable distribution. Recall that for only three cases of α there exist closed formulae for the distribution and density functions. The log-likelihood function is given by

$$\ell(\boldsymbol{\eta} | Z_{1,h}, \cdots, Z_{N-1,h}) = \sum_{k=1}^{N-1} \ln(f(Z_{k,h} | \boldsymbol{\eta})).$$

By numerical optimization of function $\ell(\cdot)$, we obtain the maximum likelihood estimator $\hat{\boldsymbol{\eta}} = (\hat{\alpha}, \hat{\sigma}, \hat{\theta})'$. In the simulations, the θ parameter is assumed to be in the set $\{1, 2\}$. In this work, the parameters of the α-stable distribution are $\alpha \in \{1.1, 1.5, 2\}$ and $\sigma = 1$. Notice that we are including the Gaussian case ($\alpha = 2$). The Monte Carlo simulation study is based on time series of samples of size $N = 2000$ derived from OU processes. We perform 500 replications for each experiment. To generate the time series we apply the discrete equation given in (4.1) and the discretization step size is considered to be 1.



(a) $\hat{\alpha}$; $\alpha = 1.1$ (b) $\hat{\sigma}$; $\alpha = 1.1$ (c) $\hat{\theta}$; $\alpha = 1.1$

(d) $\hat{\alpha}$; $\alpha = 1.5$ (e) $\hat{\sigma}$; $\alpha = 1.5$ (f) $\hat{\theta}$; $\alpha = 1.5$

(g) $\hat{\alpha}$; $\alpha = 2$ (h) $\hat{\sigma}$; $\alpha = 2$ (i) $\hat{\theta}$; $\alpha = 2$

Figure 5.1: Estimation results for OU processes when $h = 1$ and $N = 2000$. Each panel shows the results for $\theta \in \{1, 2\}$. The black lines are the medians for each experiment, the colored lines show the true parameter values, and the colored dots are the sample mean values.

Figure 5.1 shows boxplots of the parameter estimation procedure. The boxplots present the mean, median, outlier points and an idea of the variability. From these graphs we conclude that the maximum likelihood estimation performs relatively well. Comparing all graphs, it is possible to see that the worst performance occurs when $\alpha = 1.1$. In this case, there are several outlier points, and the mean estimated values are not so close to the real parameter values. This result was expected by the fact that the α parameter is too close to the range where even the first moment of the α-stable process is infinite.

The small bias values presented in Figure 5.1 indicate that, for all parameters, the mean estimated value is very close to the true parameter one. As expected, the estimator performance improves as the α parameter gets closer to $\alpha = 2$ (Gaussian case). From the graphs it is also clear that the θ parameter is better estimated than the others. There is a small difference in the results when $a = 1$ or $a = 2$: in the latter case the variability slightly increases.

From a large Monte Carlo simulation study done before, it is clear that there is no significant difference in the results for both cases $h = 1$ and $h = 0.1$, when we kept fixed the sample size N. However, when the product Nh increases, the parameter estimation improves. Another interesting characteristic is that, when the σ parameter value increases, its estimation degrades by increasing the bias value and its variability. Similar results are obtained when the θ parameter value increases. We performed an extensive simulation study and the results are available upon request.

5.2 Maximum Likelihood in the Cosine Process

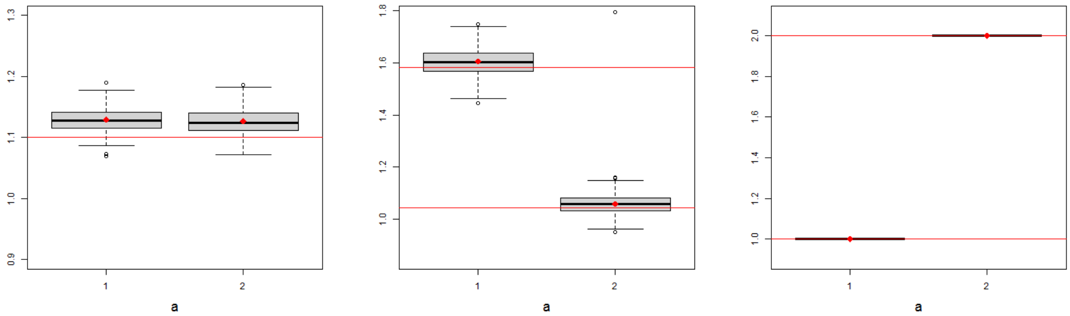
In this subsection, we estimate the parameters for the case when $\rho(t) = cos(at)$, that is, the Cosine process, via maximum likelihood procedure. From expression (4.7), notice that we can consider the Cosine process as a non-stationary AR(2) process. Let $\boldsymbol{\eta} = (\alpha, \sigma_\varepsilon, a)'$ be the parameter vector to be estimated and let $\{V_{kh}\}_{k=0}^{N-1}$ be a sample of size N of the process given by (4.6). The procedure to obtain the likelihood function is very similar to the case presented in Section 5.1.

In the simulations, the a parameter is assumed to be in the set $\{1, 2\}$. In this work, the α parameter of the α-stable distribution is in the set $\{1.1, 1.5, 2\}$ and σ_ε depends on α and a parameters (see expression (4.8)). Notice that we are including the Gaussian case ($\alpha = 2$). The Monte Carlo simulation study is based on time series of samples of size $N = 2000$ derived from Cosine processes. We perform 500 replications for each experiment. To generate the time series we apply the discrete equation given in (4.7) with discretization step size $h = 1$.

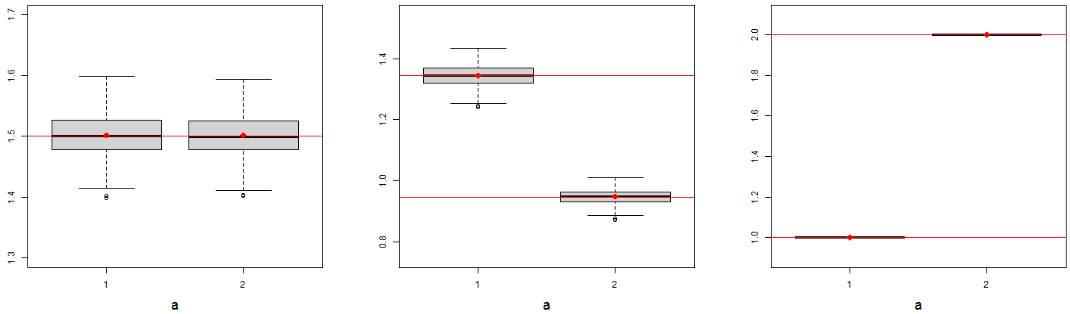
Figure 5.2 presents the results for the mle procedure. We observe that the estimation of the a parameter is very accurate, but this does not occur for the other parameters. However, when $\alpha = 2$ the estimation improves for all parameters. Comparing all graphs, the worst performance occurs when $\alpha = 1.1$, since the bias is slightly larger than the other results. There is no significant difference in the results for both cases $a = 1$ and $a = 2$. The estimation via maximum likelihood for the Cosine process was very satisfactory as for the OU process.

6 Conclusions

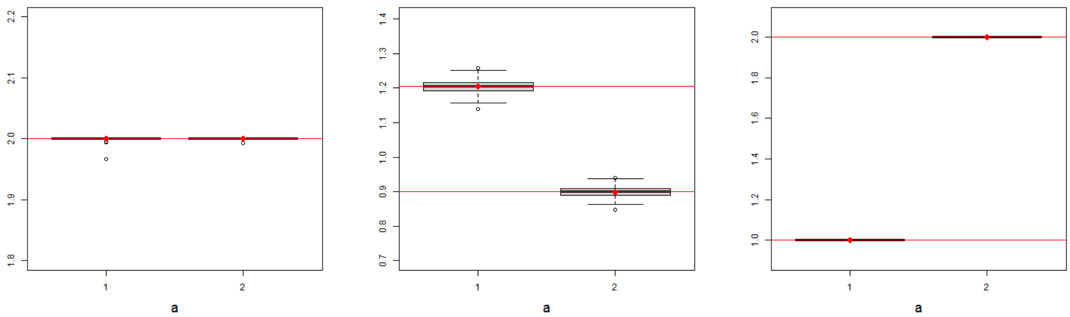
In this work we present the codifference and the spectral covariance functions as two dependence measures for the studied stochastic processes. These measures replace the autocovariance function when it is not well defined. The consistency property of the empirical codifference estimator, proposed in Subsection 2.1, is shown for stationary symmetric α-stable processes. Moreover, we present a continuous-time process arising from the generalized Langevin equation and show some of its properties. Results for the theoretical codifference and the spectral covariance functions considering the mentioned process are presented. In addition, several particular examples are discussed, showing their codifference and spectral covariance functions.



(a) $\hat{\alpha}$; $\alpha = 1.1$ (b) $\hat{\sigma}_\varepsilon$; $\alpha = 1.1$ (c) $\hat{a}$; $\alpha = 1.1$



(d) $\hat{\alpha}$; $\alpha = 1.5$ (e) $\hat{\sigma}_\varepsilon$; $\alpha = 1.5$ (f) $\hat{a}$; $\alpha = 1.5$



(g) $\hat{\alpha}$; $\alpha = 2$ (h) $\hat{\sigma}_\varepsilon$; $\alpha = 2$ (i) $\hat{a}$; $\alpha = 2$

Figure 5.2: Estimation results for Cosine processes when $h = 1$ and $N = 2000$. Each panel shows the results for $a \in \{1, 2\}$. The black lines are the medians for each experiment, the colored lines show the true parameter values, and the colored dots are the sample mean values.

Furthermore, we show via Monte Carlo simulation that maximum likelihood estimators for OU and Cosine processes present features like low bias and low variability. This simulation study shows that the mle estimation present large variability and some outlier values when $\alpha = 1.1$. This is due to the fact that the α parameter is close to the range where the first moment of the process is infinite ($0 < \alpha \leq 1$). The estimator performance improves when the α parameter gets closer to the Gaussian case ($\alpha = 2$). The Monte Carlo simulations also show that the discretization step size h of the process did not significantly matter for the OU parameters estimation. For future work we shall investigate the maximum likelihood method for other functions $\rho(\cdot)$ in the process given by the expression (3.1).

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