An algorithm for Reduced-order State Estimation of Descriptor Systems

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Abstract

An algorithm for state estimation of descriptor systems via solution of generalized Sylvester-observer equation is proposed. The algorithm is structure preserving and rich in BLAS-3 computations and thus suitable for high performance computing. The algorithm is a generalization of the one recently proposed by Datta and Sarkissian [8] for the state-estimation of standard first-order state space systems. The algorithm can easily be adapted to estimate the states of a matrix second-order control system.

1 Introduction

In this paper, we consider the problem of state estimation of the linear time-invariant continuous-time generalized state space system, also called *descriptor system*

$$\begin{array}{l} E\dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{array} \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$ and $rank(E) \geq n - r$.

If E is nonsingular, the system (1) has the standard first-order state space form

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t). \end{array} \tag{2}$$

Descriptor systems arise, for example, in modeling interconnected systems, constrained mechanical systems and economic processes [15, 22].

Furthermore, a control system represented by a system of matrix second-order differential equations of the form:

$$\begin{array}{l} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t) \\ y(t) = C_1 q(t) + C_2 \dot{q}(t) \end{array} \tag{3}$$

arising in stabilization and control of vibrating structures such as bridges, highways, automobiles and aircrafts, can be casted into the descriptor system form (in the case M is nonsingular):

$$\begin{array}{l} \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} M & \\ & -K \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} C_2 & C_1 \end{bmatrix} \begin{bmatrix} \dot{q} \\ q \end{bmatrix} \end{array} \tag{4}$$

There exist in the literature a few numerical algorithms [16, 17, 20, 21, 22] for descriptor systems. However, an algorithm for state estimation of descriptor systems was not proposed before.

There are two standard approaches for state estimation of the system (2): the *eigenvalue assignment* approach and the *Sylvester-observer equation* approach [4, 9].

The *Sylvester-observer equation approach* for state estimation of the system (2) is based on numerically solving the matrix equation

$$XA - FX = GC \tag{5}$$

which is a variation of the well known Sylvester equation $XA - BX = C$. Equation (5) is called **Sylvester-observer** equation [2, 7, 9]. In this equation, A and C are known and X, F and G are to be found. If F is a stable matrix and $P = XB$, then it can be shown [4, 9] that $e(t) = z(t) - Xx(t) \to 0$ as $t \to \infty$, where $z(t)$ satisfies the

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system of differential equations:

$$\dot{z}(t) = Fz(t) + Gy(t) + Pu(t) \ , \ z(0) = 0. \quad (6)$$

Here $y(t)$ is the observed output of the system for a chosen input $u(t)$. An estimate $\hat{x}(t)$ to the state vector $x(t)$ can then be computed [13, 14] by solving

$$\begin{bmatrix} C \\ X \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}. \quad (7)$$

It is well known that the solution X of (5) is unique if and only if $\Omega(A) \bigcap \Omega(F) = \emptyset$, where $\Omega(M)$ denotes the spectrum set of a matrix M. In order to solve (7), the computed matrix X must have full rank. For this to happen, two necessary conditions must be imposed: the pair (A, C) must be observable and the pair (F, G) must be controllable [18].

The existing numerical methods for solving the Sylvester-observer equation include :

- the recursive method by Van Dooren [19]
- the block generalization of Van Dooren's algorithm by Carvalho and Datta [3]
- the SVD-based algorithm by Datta and Sarkissian [8]
- the large-scale algorithms by Datta and Saad [6] and Calvetti and Reichel [2]
- the parallel algorithm by Bischof, Datta and Purkayastha [1].

In this paper we propose a Sylvester-observer equation approach for state estimation of the descriptor system (1). We first prove a theorem (Theorem 2.1) analogous to the Observer Theorem [4, 8] for state estimation of the standard state space system (2), showing how the *generalized Sylvester-observer equation* can be used to estimate the states of the descriptor system (1).

An SVD-based block algorithm is then proposed to solve this generalized Sylvester-observer equation. The proposed algorithm generalizes a recent SVD-based algorithm of Datta and Sarkissian [8]. The proposed algorithm is structure preserving and rich in BLAS-3 computations, thus suitable for high performance computing using the well-known package LAPACK [10].

2 State Estimation of Descriptor Systems

The analogue of the Sylvester-observer equation (5) for the descriptor system (1) is

$$XA - FXE = GC. \quad (8)$$

In (8) the matrices $E, A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{r\times n}$ are given and the matrices $X \in \mathbb{R}^{(n-r)\times n}, F \in \mathbb{R}^{(n-r)\times(n-r)}$ and $G \in \mathbb{R}^{(n-r)\times r}$ are to be found.

The *observer* system

$$\dot{z}(t) = Fz(t) + Gy(t) + XBu(t) \ , \ z(0) = 0 \quad (9)$$

now replaces (6) and an estimate $\hat{x}(t)$ is computed by solving

$$\begin{bmatrix} C \\ XE \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \quad (10)$$

where again $y(t) \in \mathbb{R}^r$ is the measured output corresponding to a chosen input $u(t) \in \mathbb{R}^m$.

In order to solve (9), the matrix XE must have full rank. Therefore the solution matrix X of (8) is desired to have full rank $n-r$ and we must assume $rank(E) \geq n-r$. In this context, it is noted that necessary conditions for a unique solution X of (8) have full rank are :

(i) The system (1) is observable

(ii) The pair (F, G) is controllable.

Also, the uniqueness of the solution X of (8) will hold if and only if $\Omega(A, E) \bigcap \Omega(F) = \emptyset$, where $\Omega(A, E)$ denotes the set of generalized eigenvalues of the pencil $P(\lambda) = A - \lambda E$ [5]. Thus, these will be our standard assumptions for the rest of the paper.

The following theorem shows that the solution $z(t)$ of (9) asymptotically approximates a linear transformation of the state vector $x(t)$ of (1) when $X \in \mathbb{R}^{s\times n}$ is the solution of (8) and matrix F is chosen to be stable (that is, all the eigenvalues have negative real parts).

Theorem 2.1: Reduced-order Observer Theorem for Descriptor systems.

The system (9) is a state-observer of the system (1), that is, $z(t)$ is an estimate of $XEx(t)$ in the sense that the error $e(t) = z(t) - XEx(t)$ approaches zero as time t increases, for any initial condition $x(0)$ and $z(0)$ if

(i) $XA - FXE = GC$

(ii) F is stable .

Proof: Using (1) and (8) we can write

$$\dot{e}(t) = \dot{z}(t) - XE\dot{x}(t) = Fz(t) + \\ +GCx(t) + XBu(t) - XAx(t) - XBu(t) = \\ = Fz(t) - FXEx(t) = F[z(t) - XEx(t)] \ .$$

Therefore $e(t) = \exp(Ft)[z(0) - XEx(0)]$, where $z(0)$ and $x(0)$ are any initial conditions. Thus $e(t) \to 0$ as $t \to \infty$, since F is stable.

3 An SVD-based Method for solving $XA - FXE = GC$

In this section, we show how the algorithm presented by Datta and Sarkissian [8] can be generalized to solve the descriptor Sylvester-observer equation (5). We recall that $E, A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{r\times n}$ are such that the system (1) is observable.

Let $\mathcal{S}$ be a self-conjugate set of complex numbers such that $\Omega(A, E) \bigcap \mathcal{S} = \emptyset$. In order to solve (5) block-wise, partition the matrix F in $q \times q$ blocks,

$$F = \begin{bmatrix} F_{11} & F_{12} & & \\ & F_{22} & F_{23} & \\ & & \dots & \dots \\ & & & F_{qq} \end{bmatrix} \tag{11}$$

where diagonal blocks $F_{ii}, i = 1, \dots, q$ of F are chosen to be matrices in real Schur form [9, 11] such that the requirement $\Omega(F) = \mathcal{S}$ is satisfied. Partitioning matrices X and G accordingly with the partition of F,

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \dots \\ X_q \end{bmatrix} \ , \ G = \begin{bmatrix} 0 \\ \dots \\ 0 \\ G_q \end{bmatrix} . \tag{12}$$

Using (11) and (12) in (8) and equating the corresponding blocks on each side, we obtain

$$X_iA - F_{ii}X_iE = F_{i,i+1}X_{i+1}E \ , \ i = 1, \dots, q-1 \tag{13}$$
$$X_qA - F_{qq}X_qE = G_qC. \tag{14}$$

Let $\tilde{F}_{qq}$ be a matrix with a prescribed spectrum and let Y_q satisfy the Sylvester equation

$$Y_qA - \tilde{F}_{qq}Y_qE = C \tag{15}$$

Taking now the SVD of Y_q^T: $Y_q^T = U_q\Sigma_qV_q^T$ gives

$$V_q\Sigma_qU_q^TA - \tilde{F}_{qq}V_q\Sigma_qU_q^TE = C.$$

If Y_q has full rank, we have

$$U_q^TA - (\Sigma_q^{-1}V_q^T\tilde{F}_{qq}V_q\Sigma_q)U_q^TE = \Sigma_q^{-1}V_q^TC \tag{16}$$

meaning that we have a solution of (14) with

$$X_q = U_q^T \ , \ F_{qq} = \Sigma_q^{-1}V_q^T\tilde{F}_{qq}V_q\Sigma_q \ , \ G_q = \Sigma_q^{-1}V_q^T. \tag{17}$$

This strategy can be repeated for each $i = q-1, \dots, 2, 1$ to obtain a solution of (13) as follows: Choosing $\tilde{F}_{ii}$ with a prescribed spectrum, we solve

$$Y_iA - \tilde{F}_{ii}Y_iE = X_{i+1}E \tag{18}$$

for Y_i and compute the SVD of Y_i^T : $Y_i^T = U_i\Sigma_iV_i^T$. If Y_i has full rank, compute

$$X_i = U_i^T \ , \ F_{ii} = \Sigma_i^{-1}V_i^T\tilde{F}_{ii}V_i\Sigma_i \ , \ F_{i,i+1} = \Sigma_i^{-1}V_i^T. \tag{19}$$

The above discussion leads to the following algorithm:

Algorithm 3.1 *: Block Algorithm for the generalized Sylvester-observer equation*

Input: Matrices $E \in \mathbb{R}^{n\times n}, A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{r\times n}$ such that the system (1) is observable and C has full rank. Self conjugate vector $\mathcal{S} \in \mathbb{C}^{n-r}$ such that $\Omega(A, E) \bigcap \mathcal{S} = \emptyset$.

Output: matrices $X \in \mathbb{R}^{(n-r)\times n}$, $F \in \mathbb{R}^{(n-r)\times(n-r)}$ and $G \in \mathbb{R}^{(n-r)\times r}$ such that $XA - FXE = GC$ and $\Omega(F) = \mathcal{S}$.

Step 1: Distribute the elements of $\mathcal{S}$ in self-conjugate groups $S_i, i = 1, \dots, q$.

Step 2: For $i = q, q-1, \dots, 1$ do Steps 3-9

Step 3: If $i = q$ set $J = C$; else set $J = X_{i+1}E$.

Step 4: Choose $\tilde{F}_{ii}$ to be a matrix in real Schur form such that $\Omega(\tilde{F}_{ii}) = S_i$.

Step 5: Solve, for Y_i, the generalized Sylvester equation $Y_iA - \tilde{F}_{ii}Y_iE = J$ using available standard software.

Step 6: From the economy size SVD of Y_i^T, compute matrices U_i and V_i and diagonal matrix Σ_i such that $Y_i^T = U_i \Sigma_i V_i^T$.

Step 7: If Σ_i is nonsingular, compute

$$F_{ii} = \Sigma_i^{-1} V_i^T \tilde{F}_{ii} V_i \Sigma_i$$

taking advantage of the structure of $\tilde{F}_{ii}$. Otherwise, find another set S_i and go back to Step 5.

Step 8: If $i = q$, compute $G_q = \Sigma_q^{-1} V_q^T$; else, compute $F_{i,i+1} = \Sigma_i^{-1} V_i^T$.

Step 9: Set $X_i = U_i^T$.

Step 10: Form matrices X, F and G, from their blocks computed above.

Remarks

1. As in [8], the above algorithm above produces a matrix X such that every block-row $X_i, i = 1, \ldots, q$ is a matrix whose rows form an orthonormal set of vectors.

2. The algorithm does not require reduction of the system matrices E, A and C. This is specially attractive when E and A are large and sparse, as long as we are able to exploit this structure in the solution of the subproblems in Step 5.

3. If the matrices E and A are dense, then the pair (E, A) can be reduced to (T, H), where H is a block upper Hessenberg matrix and T is upper triangular. The advantage of this block hessenberg-triangular form can again be exploited in the solution of the subproblems in Step 5.

4. The algorithm is rich in BLAS-3 computations and thus is suitable for high-performance computing using LAPACK [10] .

4 Reduced-order State Estimation for Matrix Second-order Systems

The results of the previous section can be conveniently adapted to estimate the states of the second-order model (3). To do this, we recall the first-order representation (4). Suppose that the matrices $F \in \mathbb{R}^{(2n-r)\times(2n-r)}$, $X \in \mathbb{R}^{(2n-r)\times 2n}$, $G \in \mathbb{R}^{(2n-r)\times r}$ have been computed such that F is a stable matrix and

$$X \begin{bmatrix} M & \\ & -K \end{bmatrix} - FX \begin{bmatrix} 0 & M \\ M & D \end{bmatrix} = G \begin{bmatrix} C_2 & C_1 \end{bmatrix}. \tag{20}$$

After partitioning $E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}$ and $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, approximations $\hat{x}(t)$ and $d(t)$ of $x(t)$ and $\dot{x}(t)$ can be found by solving

$$\begin{bmatrix} C_2 & C_1 \\ XE_1 & XE_2 \end{bmatrix} \begin{bmatrix} d(t) \\ \hat{x}(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \tag{21}$$

where $z(t) \in \mathbb{R}^{2n-r}$ is the solution of the differential equation

$$\dot{z}(t) = Fz(t) + Gy(t) + X_2 Bu(t) \tag{22}$$

satisfying any initial condition $z(0) = z_0$.

Algorithm 4.1: State Estimation of Matrix Second-order Systems.

Input: System matrices M, D, K, B, C_1 and C_2 of the system (3), assumed to be observable, where C_1 and C_2 are matrices with full rank.

Output: Estimates $\hat{x}(t)$ and $d(t)$ of $q(t)$ and $\dot{q}(t)$ respectively.

Step 1: Solve the descriptor Sylvester-observer equation (20).

Step 2: Partition $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$ and solve the differential equation (22).

Step 3: Compute $\hat{x}(t)$ and $d(t)$ by solving (21).

5 Numerical Examples

Example 1: Consider the second-order system (3) with

$$M = \begin{bmatrix} 2.9550 & & \\ & 3.5220 & \\ & & 0.5710 \end{bmatrix}$$

$$D = \begin{bmatrix} 193.4974 & -1.1174 & 0 \\ -1.1174 & 40.1125 & -2.6801 \\ 0 & -2.6801 & 50.1971 \end{bmatrix}$$

$$K = \begin{bmatrix} 382.3000 & -382.3000 & 0 \\ -382.3000 & 432.8500 & -50.5500 \\ 0 & 50.5500 & -50.5500 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1.9200 & 19.8400 & 32.7200 \\ 53.3700 & 56.0300 & 84.1100 \end{bmatrix}.$$

Solving the generalized Sylvester-observer equation (20) using Algorithm 3.1, we obtain

$$X = \begin{bmatrix} .0003 & -.9960 & .0664 & .0353 & .0206 & .0427 \\ -.8329 & -.0187 & -.3331 & .1653 & -.3858 & .1370 \\ -.3052 & .4943 & -.8086 & .0299 & -.0460 & .0747 \\ -.9060 & .0215 & .3763 & .1083 & -.0931 & .1289 \end{bmatrix}$$

$$F = \begin{bmatrix} -10.0000 & 1.6329 & 4.3713 & -0.4391 \\ -61.2406 & -10.0000 & 2.6890 & 26.7700 \\ 0 & 0 & -11.3836 & 0.3206 \\ 0 & 0 & 37.1845 & -18.6164 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.1568 & 0.3912 \\ -4.2136 & 1.6884 \end{bmatrix}$$

where F is a stable matrix with chosen eigenvalues
$\mathcal{S} = \{\ -10 \quad -10-10i \quad -10+10i \quad -20\ \}$

Figure 1 below shows a comparison between the third components of the actual state $\dot{q}(t)$, computed by directly solving (4) numerically, and the estimated derivative $d(t)$, computed using (21) where $z(t)$, given by (22), was computed using **Matlab**'s **ode23** package . The input vector $u(t)$ and the initial conditions where chosen as: $u(t) = \delta(t)[1;1;1]$ (the Dirac Delta) and $z(0^-) = x(0^-) = 0$.

The plot shows that the third component of $d(t)$, the estimated derivative, approaches the third component of the actual state derivative $\dot{q}(t)$ as time t increases. Similar plots can be drawn for the other components of $d(t)$ and $\hat{x}(t)$.

Example 2: In order to illustrate the solution of the generalized Sylvester-observer equation in the case that E is singular, consider

$$E = diag\begin{pmatrix} 8 & 6 & 4 & 2 & 1 & 0 \end{pmatrix}$$

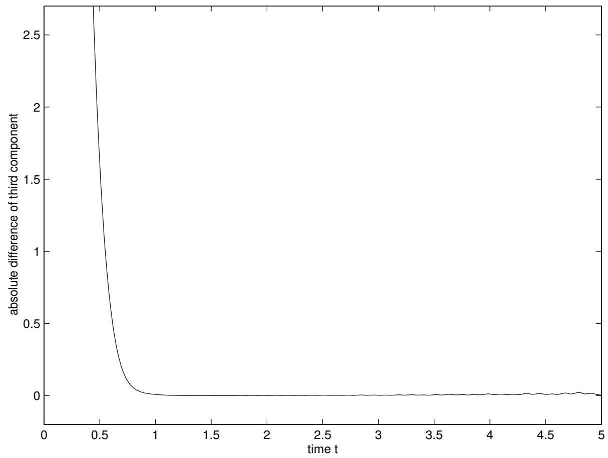

Figure 1: Plot of the absolute difference between the third components of $d(t)$ and $\dot{q}(t)$.

$$A = \begin{bmatrix} 0.45 & -0.04 & 0.42 & -0.09 & -0.36 & -0.48 \\ -0.27 & -0.48 & 0.24 & 0.39 & -0.30 & 0.25 \\ 0.11 & 0.32 & -0.32 & -0.44 & -0.30 & -0.05 \\ -0.01 & -0.06 & -0.09 & -0.15 & 0.10 & 0.43 \\ 0.39 & 0.12 & 0.44 & 0.31 & -0.23 & -0.03 \\ 0.25 & 0.30 & 0.42 & -0.49 & -0.30 & -0.08 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.8462 & 0.2026 & 0.8381 & 0.6813 & 0.8318 & 0.7095 \\ 0.5252 & 0.6721 & 0.0196 & 0.3795 & 0.5028 & 0.4289 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where matrix A is nonsingular but $rank(E) = 5$. Solving equation (8), using Algorithm 3.1, we obtain

$$X = \begin{bmatrix} .0345 & .0610 & .0811 & -.1795 & -.1976 & -.9577 \\ .0953 & .6812 & -.5731 & -.3357 & .2928 & .0008 \\ .0071 & .0100 & .0200 & -.0353 & -.0369 & -.9984 \\ .1220 & .2251 & .2725 & -.6288 & -.6794 & .0559 \end{bmatrix},$$

$$F = \begin{bmatrix} -4.0000 & 0.0260 & 1.1930 & 0.7271 \\ -616.2763 & -4.0000 & -112.0226 & 183.8075 \\ 0 & 0 & -6.6134 & 0.0387 \\ 0 & 0 & 93.6387 & -5.3866 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.0625 & 0.0456 \\ -2.2408 & 3.0764 \end{bmatrix}$$

where the matrix F is stable with spectrum
$$\mathcal{S} = \{ \; -4 \quad -4-4i \quad -4+4i \quad -8 \; \} .$$

6 Conclusions

Descriptor systems arise in many practical applications. Although numerical methods for control problems in standard state-space systems are well developed, only a few such algorithms have so far been extended to solutions of corresponding descriptor problems. There is certainly a need for more numerically viable algorithms for descriptor control problems.

This paper proposes a numerical algorithm for state-estimation of time-invariant continuous-time descriptor system. The algorithm is developed by generalizing the Sylvester equation approach for the problem in standard state-space systems.

As a by-product of this algorithm, an algorithm for state-estimation of continuous-time matrix second-orde system is obtained. The algorithm is rich in BLAS-3 (Basic Linear Algebra Subroutines-3) computations and is thus suitable for high-performance computing by taking advantage of the software package LAPACK.

Investigation into certain theoretical, computational and implementational issues, including numerical robustness of the algorithm and implementation on high-performance computers using large-scale real-life data is in order.

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