A New Block Algorithm for Solving the Sylvester-observer Equation.

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Abstract

A new block algorithm for computing a full rank solution of the Sylvester-observer equation is proposed. The algorithm is structure preserving and rich in BLAS-3 computations and thus suitable for high performance computing. Furthermore, the results on numerical experiments on benchmark examples show that the algorithm has better accuracy than that of some of the existing block algorithms for this problem.

1 Introduction

Consider the matrix equation

$$XA - FX = GC \tag{1}$$

where

- matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{r\times n}$ and a self-conjugate set $\mathcal{S} \in \mathbb{C}^{n-r}$ are given such that (A, C) is observable and $\Omega(A) \bigcap \mathcal{S} = \emptyset$. Here $\Omega(M)$ denotes the spectrum set of a matrix M.
- matrices $X \in \mathbb{R}^{s\times n}$, $F \in \mathbb{R}^{s\times s}$ and $G \in \mathbb{R}^{s\times r}$ are to be found such that X has full rank , $\Omega(F) = \mathcal{S}$ and (F, G) is a controllable pair.

A well-known method for solving the Sylvester-observer equation, based on the observer Hessenberg decomposition of the observable pair (A, C) is due to Van Dooren [14]. The method is recursive in nature and compute the solution matrix X and the matrices F and G recursively, one row or column at a time.

Recently three block algorithms (see [1, 3, 5]) have been proposed this problem. These block algorithms are rich in matrix-matrix computations and therefore suitable for high-performance computing [7].

In this paper, we present another block algorithm. This new algorithm seems to be more accurate than some of the block algorithms mentioned above and is generated to give a full-rank solution X with a triangular structure. This structure can be exploited in solving the system (3).

The problem of solving (1) arises in the construction of reduced order Observers (see [11]) for the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Du(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2}$$

in the context of **State Estimation**. Once a solution triple (X, F, G) is computed, then an estimate $\hat{x}(t)$ to the state vector $x(t)$ can be computed by solving the following algebraic system of equations (see [11, 12]):

$$\begin{bmatrix} C \\ X \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}. \tag{3}$$

Here $z(t)$ is a solution of

$$\dot{z}(t) = Fz(t) + Gy(t) + XDu(t) \tag{4}$$

where it is assumed that we have measured the output $y(t)$ for a given input $u(t)$ in some large time interval.

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2 A New Block Algorithm

We propose to solve (1) by imposing some structure on the right hand side of the equation. This means that (like in the SVD-based method [5]) no reduction is imposed on the system matrix. Being more specific, we will construct a solution of

$$XA - FX = R \tag{5}$$

satisfying $\Omega(F) = \mathcal{S}$ for some matrix R , to be constructed , for which we are sure to be able to solve $GC = R$ for $G \in \mathbb{R}^{s \times r}$ later. As the solution X is being computed, a Householder-QR [8] based strategy will reshape it so that at the end of the process X is a full-rank upper triangular matrix.

First, we investigate the solution of $GC = R$ for G. Assume C has full rank r and let $C = R_c Q_c$ be the skinny RQ factorization of C (here $Q_c \in \mathbb{R}^{r \times n}$ and $R_c \in \mathbb{R}^{r \times r}$) (see [8]). If we choose

$$R = \begin{bmatrix} N_1 \\ N_2 \\ \cdots \\ N_p \end{bmatrix} Q_c \tag{6}$$

, where $N_i \in \mathbb{R}^{n_i \times r}$, $i = 1, \ldots, p$, and $n_1 + n_2 + \ldots + n_p = s$, then we can find a solution $G \in \mathbb{R}^{s \times r}$ of $GC = R$ where $G_i R_c = N_i$, $i = 1, \ldots, p$ and

$$G = \begin{bmatrix} G_1 \\ G_2 \\ \cdots \\ G_p \end{bmatrix}. \tag{7}$$

In particular, the choice $N_1 = I_r$ ensures that $rank(R) = rank(C) = r$.

Second, we partition

$$X = \begin{bmatrix} X_1 \\ \cdots \\ X_p \end{bmatrix} \quad , \; F = \begin{bmatrix} F_{11} & & \\ F_{12} & F_{22} & \\ F_{p1} & F_{p,p-1} & F_{pp} \end{bmatrix}. \tag{8}$$

Substituting (6) and (8) into (5) and equating corresponding blocks on the right and left hand sides of (5), we obtain

$$X_1 A - F_{11} X_1 = N_1 Q_c \tag{9}$$

$$X_i A - F_{ii} X_i = N_i Q_c + \sum_{j=1}^{i-1} F_{ij} X_j, i = 2, \ldots, p. \tag{10}$$

Therefore, as long as the elements of the given vector $\mathcal{S}$ can be successfully distributed in self-conjugate groups $S_i \in \mathbb{C}^{n_i}$, $i = 1, \ldots, p$, to be assigned as eigenvalues of the block matrices F_{ii}, $i = 1, \ldots, p$, we are able to construct matrices X, F and G from their blocks computed recursively using (9) and (10).

After each sub-matrix X_i of the solution X has been computed, an orthogonal transformation is applied such that the upper triangular structure is created. This upper triangular structure can be exploited in solving the system (3).

Therefore, a step-by-step QR factorization will implicitly be performed on the matrix X. Assume that in the $i-th$ stage of this process all the blocks $X_j, j = 1, \ldots, i-1$ form an upper triangular matrix and that the block X_i has just been computed. Placing X_i in the bottom of this matrix of X_j's gives an augmented matrix like

$$\begin{bmatrix} X_1 \\ \cdots \\ X_{i-1} \\ X_i \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * & * & * & * & * \\ & * & * & * & * & * & * & * & * & * \\ & & * & * & * & * & * & * & * & * \\ & & & & & & \cdots & & \cdots & \\ & & & & * & * & * & * & * & * \\ & & & & & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * \\ & & & \cdots & & & \cdots & & & \\ * & * & * & * & * & * & * & * & * & * \end{bmatrix}$$

which can be reduced to an upper triangular form by orthogonal transformation applied to the its rows, that is, via left multiplication by a sequence of Householder matrices, for instance. This means that, after the reduction process, the matrices $X_1, \ldots, X_{i-1}$ will also be updated by

$$X_i^{new} = Q_i^T X_i, i = 1, 2, \ldots$$

where Q_i is an orthogonal matrix. The matrix equation

$$\begin{bmatrix} X_1 \\ \cdots \\ X_{i-1} \\ X_i \end{bmatrix} A - F^i \begin{bmatrix} X_1 \\ \cdots \\ X_{i-1} \\ X_i \end{bmatrix} = \begin{bmatrix} G_1 \\ \cdots \\ G_{i-1} \\ G_i \end{bmatrix} C \tag{11}$$

where

$$F^i = \begin{bmatrix} F_{11} & \dots & F_{1,i-1} & 0 \\ \dots & \dots & \dots & 0 \\ F_{i-1,1} & & F_{i-1,i-1} & 0 \\ F_{i1} & \dots & \dots & F_{ii} \end{bmatrix} \tag{12}$$

is also updated at the same time as follows:

$$Q_i^T \begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix} A - Q_i^T F^i Q_i \cdot Q_i^T \begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix} = Q_i^T \begin{bmatrix} G_1 \\ \dots \\ G_{i-1} \\ G_i \end{bmatrix} C$$

and

$$\begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix}^{new} A - (F^i)^{new} \begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix}^{new} = \begin{bmatrix} G_1 \\ \dots \\ G_{i-1} \\ G_i \end{bmatrix}^{new} C \tag{13}$$

meaning that it is possible to update the solution matrices, at every step of the orthogonal reduction, simply by computing

$$(F^i)^{new} = Q_i^T F^i Q_i \tag{14}$$

$$\begin{bmatrix} G_1 \\ \dots \\ G_{i-1} \\ G_i \end{bmatrix}^{new} = Q_i^T \begin{bmatrix} G_1 \\ \dots \\ G_{i-1} \\ G_i \end{bmatrix}. \tag{15}$$

3 An algorithm for solving $XA - FX - GC$

Based on the above discussion, the following algorithm for the Sylvester-observer equation is proposed.

Input: Matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{r\times n}$ such that (A, C) is observable and C has full rank. Self-conjugate vector $\mathcal{S} \in \mathbb{C}^{n-r}$ such that $\mathcal{S} \bigcap \Omega(A) = \emptyset$.

Output: Matrices $X \in \mathbb{R}^{n-r\times n}$ upper triangular, $F \in \mathbb{R}^{(n-r)\times(n-r)}$ and $G \in \mathbb{R}^{(n-r)\times r}$ such that

$$XA - FX = GC \text{ and } \Omega(F) = \mathcal{S} .$$

Step 1: Let $l = r$, $N_1 = I_{r\times r}$.

Step 2: Compute the thin RQ factorization of C : $R_c Q_c = C$ where $Q_c \in \mathbb{R}^{r\times n}$ and $R_c \in \mathbb{R}^{r\times r}$.

Step 3: For $i = 1, 2, \dots$ do

Step 4: Set S_i be the next self-conjugate subset of $\mathcal{S}$ having l elements.

Step 5: Set $F_{ii} \in \mathbb{R}^{l\times l}$ to be any matrix in upper Real Schur form verifying $\Omega(F_{ii}) = S_i$.

Step 6: Free assignment step. If $i > 1$ set $N_i \in \mathbb{R}^{l\times s_i}$ and $F_{ij} \in \mathbb{R}^{l\times s_j}$, $j = 1, \dots, i-1$ to be arbitrary matrices.

Step 7: Solve the Sylvester equation

$$X_i A - F_{ii} X_i = N_i Q_c + \sum_{j=1}^{i-1} F_{ij} X_j \tag{16}$$

for $X_i \in \mathbb{R}^{l\times n}$.

Step 8: Set n_i to be the number of rows of X_i that are linearly independent of the rows of the matrix $\begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \end{bmatrix}$.

If $n_i < l$ then set $l = n_i$, restore S_i to $\mathcal{S}$ and do Steps 4 - 8 again.

Else, set $l = n_i$.

Step 9: Find, implicitly, an orthogonal matrix Q_i that reduces the matrix

$$\begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix}$$

to upper triangular form via left multiplication by Q_i^T. Then compute the updates

$$\begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix} \leftarrow Q_i^T \begin{bmatrix} X_1 \\ \dots \\ X_{i-1} \\ X_i \end{bmatrix},$$

$$\begin{bmatrix} G_1 \\ \dots \\ G_{i-1} \\ G_i \end{bmatrix} \leftarrow Q_i^T \begin{bmatrix} G_1 \\ \dots \\ G_{i-1} \\ N_i R_c^{-1} \end{bmatrix},$$

$$F^i = Q_i^T F^i Q_i$$

Step 10: If $n_1 + \ldots + n_i = s$ then let $p = i$ and exit loop.

End For

Step 11: Form matrices X, F and G from their blocks computed above.

Remarks:

1. Some compatibility between the structure of the vector $\mathcal{S}$ and the parameters n_i, $i = 1, \ldots, p$ is required so that Step 4 is always possible to be accomplished.
2. In Step 6, a random number generator can be used. It is also possible to exploit the freedom to assign F_{ij} to facilitate the solution of the Sylvester equation in Step 7, specially if A is large and sparse, situation in which iterative procedures are almost unavoidable.
3. If matrix A is small and dense, an orthogonal similarity reduction

$$A \leftarrow P^T AP, C \leftarrow CP$$

can be used so to bring Hessenberg structure to matrix A. This allows Step 7 to be computed efficiently so that the whole algorithm requires order n^3 flops. If (X_h, F, G) is the solution of this reduced problem, $X = X_h P^T$ recovers the solution of the original problem.

4. The algorithm is rich in BLAS-3 computations and thus is suitable for high-performance computing using LAPACK [7].

4 Conclusion

A new block algorithm for solving the Sylvester-observer equation is proposed. The algorithm does not require the reduction of the system matrix A as long as the solution of small sized standard Sylvester equations do not required this reduction. This algorithm is well-suited for high-performance implementation using $LAPACK$ and it seems to be accurate compared with similar ones; numerical stability properties have not been studied yet.

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