A Block Algorithm for the Sylvester-observer Equation arising in State-Estimation.

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Abstract

We propose a new algorithm for solving the Sylvester-observer equation arising in the construction of the Luenberger observer.

The algorithm is a block-generalization of Van Dooren's scalar algorithm. It is more efficient than Van Dooren's and the recent block algorithm by Datta and Sarkissian.

Furthermore, the algorithm is well-suited for implementation on some of today's powerful high performance computers using the high-quality software package LAPACK.

1 Introduction

Consider the problem of state estimation for the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$
$$y(t) = Cx(t). \tag{2}$$

A well-known approach for constructing a reduced-order observer in control theory (see, e.g, Luenberger (1966), Chen (1984), Datta(1994)) is via solution of the Reduced-Order Sylvester-Observer Equation:

$$XA - TX = GC \tag{3}$$

where the state matrix $A \in R^{n\times n}$ and the output matrix $C \in \mathbb{R}^{r\times n}$ $(r < n)$ are known and the matrices $T \in \mathbb{R}^{(n-r)\times(n-r)}$, $X \in \mathbb{R}^{(n-r)\times n}$ and $G \in \mathbb{R}^{(n-r)\times r}$ are to be found. The matrix T can be chosen to have a desired spectrum. In particular, if T is chosen to be a stable matrix, then it can be shown that the vector $z(t) \to Xx(t)$, where $z(t)$ is obtained by solving the differential equation:

$$\dot{z}(t) = Tz(t) + Gy(t) + XBu(t). \tag{4}$$

Once $z(t)$ is found, the estimated states $\hat{x}(t)$ are given by

$$\hat{x}(t) = \begin{bmatrix} C \\ X \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}. \tag{5}$$

Another important application of (3) is to the multi-input pole placement problem (Bhattacharyya and DeSouza (1982)).

A usual way to solve the equation (3) is to choose the matrices G and T conveniently and then solve for X. For example, if the matrix T is chosen as a real-Schur matrix with the given spectrum and G is chosen to be an identity matrix, then X can be found using the well-known Hessenberg-Schur (Golub, Nash and Van Loan (1979)) algorithm. However, this special choice of G may not be desirable in control applications. A novel solution method is due to Van Dooren (1984). Van Dooren's method is based on reduction of the observable pair (A, C) to an observer-Hessenberg pair $(H, \bar{C})$; that is, an orthogonal matrix P is constructed such that

$$P^T AP = H \ , \ CP = \bar{C} \tag{6}$$

where H is a block upper Hessenberg matrix, and $\bar{C} = \begin{pmatrix} 0 & \dots & 0 & C_1 \end{pmatrix}$. The reduced equation

$$YH - TY = G\bar{C} \tag{7}$$

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is then solved for Y, and finally X is recovered from Y via $X = YP^T$.

The reduction to the observer-Hessenberg form $(H, \bar{C})$ is achieved by means of the well-known staircase algorithm (Boley(1981), Van Dooren (1981), Datta (2001),etc.). The solution Y of the reduced equation (7) is computed recursively, one row at a time. The block nature of the block Hessenberg matrix H is, however, not exploited in the algorithm.

In this paper, we propose a block generalization of Van Dooren's algorithm. We show that by exploiting the block structure of H, matrices Y and G can be computed recursively in a block row-wise fashion by solving only triangular systems with multiple right hand sides. The special forms of Y and $\tilde{C}$, constructed by our procedure, ensure that the compound matrix in (5) is nonsingular.

The other recent algorithms for the Sylvester-observer equation include the large-scale Arnoldi method by Datta and Saad(1991), the parallel algorithm by Bischof, Datta and Purkayastha (1996), and the block algorithm by Datta and Sarkissian (2000) , algorithms by Tsui (1993), Calvetti, Lewis and Reichel (2001), Datta and Hetti (1997),etc.

The paper by Calvetti, Lewis and Reichel (2001) investigates into the numerical properties of the Datta-Saad algorithm (1991); in particular, it shows that the numerical stability of the later can be significantly improved if the eigenvalues of T are properly chosen.

2 Orthogonal Reduction to Observer Hessenberg form by Block Arnoldi-type method.

Given a pair of matrices (A, C), there always exists an orthogonal matrix P such that

$$P^TAP = \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1k} \\ H_{21} & H_{22} & \dots & H_{2k} \\ 0 & H_{32} & \dots & H_{3k} \\ 0 & \dots & H_{k,k-1} & H_{kk} \end{bmatrix} \quad (8)$$

$$\tilde{C} = CP = \begin{bmatrix} 0 & \dots & 0 & C_1 \end{bmatrix} \quad (9)$$

where $H_{ij} \in \mathbb{R}^{n_i \times n_j}$, $n_i = rank(n_{i,i-1})$, $i = 2, 3, \dots, k$. Matrices $H_{i,i-1}, i = 2, \dots, k$ are upper triangular and nonsingular , except possibly by $H_{k,k-1}$, which is zero if the system is unobservable. Matrix C_1 is lower triangular with respect to its anti-diagonal , and will be nonsingular iff C has full-rank.

The pair $(H, \tilde{C})$ is called an *observer Hessenberg* pair and *provides a numerically viable test for observability.* The subdiagonal blocks $H_{i,i-1}, i = 2, \dots, k$ are, in general, of different sizes. In our algorithm, to be described in the next section, we need these subdiagonal blocks to be of the same size (except, possibly one at an edge of the matrix H), and this excludes the possibility of using the stair-case algorithm for observer-Hessenberg reduction. Instead, we use the block Arnoldi method to ensure the equal size of the subdiagonal blocks of H.

The block-Arnoldi method, when applied to $A^{n \times n}$ and run for complete n steps, reduces A to a block Hessenberg matrix. For details, see Saad (1992). We adapted this procedure to reduce the pair of matrices (A^T, C^T) to a controller-Hessenberg form, from where the observer-Hessenberg form of (A, C) is easily computed. Similar procedure has been used in Carvalho and Datta (2001).

3 The Proposed Block Algorithm for the Sylvester-Observer Equation.

Let $p = \lfloor n/r \rfloor$, let $q = \lfloor s/r \rfloor$ and let $\tilde{r}$ be such that $\tilde{r} + (p-1)r = n$.

Assume that the observable pair (A, C) has been transformed to an observer-Hessenberg pair $(H, \bar{C})$ using the block Arnoldi method , that is, a block orthogonal matrix P has been computed such that

$$P^TAP = H \text{ and } CP = \begin{bmatrix} 0 & \dots & 0 & C_1 \end{bmatrix}$$

where $H = (H_{ij})$ is block upper Hessenberg with subdiagonal blocks $H_{i+1,i} \in \mathbb{R}^{r \times r}$, $\quad i = 2, 3, \dots$ and $H_{21} \in \mathbb{R}^{\tilde{r} \times r}$. Then the equation (3) reduces to (7), where $Y = XP$.

Partitioning the matrices T, Y and G conformably with H_{ij}, allows us to write (7) as

$$\begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1p} \\ & Y_{22} & \dots & Y_{2p} \\ & & Y_{qq} & Y_{qp} \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} & \dots & H_{1p} \\ H_{21} & H_{22} & \dots & H_{2p} \\ & H_{32} & \dots & H_{3p} \\ & & H_{p-1,p} & H_{pp} \end{bmatrix} -$$
$$\begin{bmatrix} T_{11} & & \\ T_{21} & T_{22} & \\ T_{q1} & \dots & T_{qq} \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1p} \\ & Y_{22} & \dots & Y_{2p} \\ & & Y_{qq} & Y_{qp} \end{bmatrix} =$$
$$\begin{bmatrix} G_1 \\ \dots \\ G_q \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 & C_1 \end{bmatrix}.$$

We set $Y_{ii} = I_{r \times r}, i = 1, 2, \dots, q$ for simplicity. Since matrix T is required to have a preassigned spectrum $\mathcal{S}$, we distribute the elements of $\mathcal{S}$ among the diagonal blocks of T in such a way that $\Omega(T) = \mathcal{S}$, where $\Omega(M)$ denotes the spectrum of M. A complex conjugate pair is distributed as a 2×2 matrix and a real one as a 1×1 scalar on the diagonal of T. Note that, some compatibility between the structure of $\mathcal{S}$ and the parameter r is required to exist for this to happen.

Equating now the corresponding blocks on left and right hand sides we obtain:

$$\sum_{k=i}^{j+1} Y_{ik} H_{kj} - \sum_{k=1}^{min(i,j)} T_{ik} Y_{kj} = 0 \qquad (10)$$
$$(j = 1, 2, \dots, p-1)$$
$$\sum_{k=i}^{p} Y_{ik} H_{kp} - \sum_{k=1}^{i} T_{ik} Y_{kp} = G_i C_1. \qquad (11)$$

From (10) and (11) , we conclude

$T_{ij} = 0$ for $j = 1, 2, \dots, i-2$, and

$T_{ij} = H_{ij}$ for $j = i-1$.

Thus, equations (10) and (11) are reduced to

$$\sum_{k=i}^{j+1} Y_{ik} H_{kj} - \sum_{k=max(i-1,i)}^{i} T_{ik} Y_{kj} = 0 \qquad (12)$$
$$(j = i, i+1, \dots, p-1)$$
$$\sum_{k=i}^{p} Y_{ik} H_{kp} - \sum_{k=max(i-1,1)}^{i} T_{ik} Y_{kp} = G_i C_1, \qquad (13)$$

for $i = 1, 2, \dots, q$. For a computational purpose we rewrite equation (12) as

$$\sum_{k=i}^{j} Y_{ik} H_{kj} + Y_{i,j+1} H_{j+1,j} - \sum_{k=max(i-1,1)}^{i} T_{ik} Y_{kj} = 0$$
$$(j = i, i+1, \dots, p-1)$$

that is, for $j = i, i+1, \dots, p-1$,

$$Y_{i,j+1} H_{j+1,j} = -\sum_{k=i}^{j} Y_{ik} H_{kj} + \sum_{k=max(i-1,1)}^{i} T_{ik} Y_{kj} \qquad (14)$$

Equations (13) and (14) allow us to compute the off-diagonal blocks Y_{ij} of Y and the blocks G_i of G recursively.

This is illustrated in the following, in the special case when $p = 4, q = 3$:

First row : $i = 1$

$H_{11} + Y_{12} H_{21} - T_{11} = 0$ (Solve for Y_{12})

$H_{12} + Y_{12} H_{22} + Y_{13} H_{32} - T_{11} Y_{12} = 0$ (Solve for Y_{13})

$H_{13} + Y_{12} H_{23} + Y_{13} H_{33} + Y_{14} H_{43} - T_{11} Y_{13} = 0$ (Solve for Y_{14})

$H_{14} + Y_{12} H_{24} + Y_{13} H_{34} + Y_{14} H_{44} - T_{11} Y_{14} = G_1 C_1$ (Solve for G_1)

Second row : $i = 2$

$H_{22} + Y_{23} H_{32} - T_{21} Y_{12} - T_{22} = 0$ (Solve for Y_{23})

$H_{23} + Y_{23} H_{33} + Y_{24} H_{43} - T_{21} Y_{13} - T_{22} Y_{23} = 0$ (Solve for Y_{24})

$H_{24} + Y_{23} H_{34} + Y_{24} H_{44} - T_{21} Y_{14} - T_{22} Y_{24} = G_2 C_1$ (Solve for G_2)

Third row : $i = 3$

$H_{33} + Y_{34} H_{43} - T_{32} Y_{23} - T_{33} = 0$ (Solve for Y_{34})

$H_{34} + Y_{34} H_{44} - T_{32} Y_{24} - T_{33} Y_{34} = G_3 C_1$ (Solve for G_3)

The above discussion leads to the following algorithm:

Algorithm 3.1: Solution of the Sylvester Observer Equation.

Input: Matrices $A \in R^{n \times n}$, $C \in \mathbb{R}^{r \times n}$ of full-rank and the self-conjugate set $\mathcal{S} \in \mathbb{C}^{n-r}$.

Output: Block matrices X,T and G, such that $\Omega(T) = \mathcal{S}$ and $XA - TX = GC$.

Step 1: Reduce (A, C) to its observer-Hessenberg form $(H, \bar{C})$ using the Block Arnoldi Method.

Step 2: Set $p = \lceil n/r \rceil$, $q = \lceil s/r \rceil$ and determine $\tilde{r}$ such that $\tilde{r} + (p-1)r = n$.

Step 3: Distribute the elements of $\mathcal{S}$ along the diagonal blocks $T_{ii}, i = 1, 2, \ldots q$ such that $\Omega(T) = \mathcal{S}$; the complex conjugate pairs as 2×2 blocks and the real ones as 1×1 scalars along the diagonal of the matrix T.

Step 4: Set $Y_{11} = I_{\tilde{r} \times \tilde{r}}$.

Step 5: For $i = 2, 3, \ldots, q$ set
$$T_{i,i-1} = H_{i,i-1}\ , \qquad Y_{ii} = I_{r \times r}$$

Step 6: For $i = 1, 2, \ldots, q$ do Steps 7 and 8

Step 7: For $j = i, i+1, \ldots, p-1$, solve the upper triangular system for $Y_{i,j+1}$:
$$Y_{i,j+1} H_{j+1,j} = -\sum_{k=i}^{j} Y_{ik} H_{kj} + \sum_{k=max(i-1,1)}^{i} T_{ik} Y_{kj}$$

Step 8: Solve the triangular system for G_i:
$$G_i C_1 = \sum_{k=i}^{p} Y_{ik} H_{kp} - \sum_{k=max(i-1,1)}^{i} T_{ik} Y_{kp}$$

Step 9: Form the block matrix Y from the computed blocks.

Step 10: Compute $X = Y P^T$.

Return

Remark: The computation in (5) requires the solution of the linear system
$$\begin{bmatrix} C \\ X \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$$
which, after using (6), can be shown to be equivalent to
$$\begin{bmatrix} \tilde{C} \\ Y \end{bmatrix} \hat{x}(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} P$$
which is nonsingular block upper Hessenberg by the construction of Y. This structure is very important from the computational point of view since it can possibly be exploited through high performance computations.

4 Flop-count and Comparison of Efficiency.

4.1 Flop count of Algorithm 3.1.

Efficiency here is meant by the number of floating point operations (flops) consumed by the algorithm. A "flop" is an arithmetic operation $+, -, \times, \div$.

In this section, we first give a flop count for the proposed algorithm and then compare the efficiency of the algorithm with that of Van Dooren and that of Datta and Sarkissian, with respect to flop count.

1. Reduction to observer Hessenberg form using the Block Arnoldi Method:
$$4n^3 + \left[\frac{k}{2} + 2r\right] n^2 \text{ flops}$$
2. Computation of Y using Steps 4 - 8 of the algorithm:
$$\sum_{i=1}^{p-1} \sum_{j=i}^{p} \left[(j-i+1)(2r^3) + 2(2r^3) + r^2\right] =$$
$$\sum_{i=1}^{p-1} \sum_{j=i}^{p} \left\{\left[2(j-i) + 7\right] r^3 + r^2\right\} \approx$$
$$\sum_{i=1}^{p-1} \left[(p-i)^2 + (p-i)\right] r^3 \approx$$
$$\approx \left[\frac{(p-1)p(2p-1)}{6} + \frac{(p-1)p}{2}\right] r^3$$
$$\approx \frac{n^3}{3} + \frac{rn^2}{2} \text{ flops.}$$
3. Computation of X from Y : n^3 flops (note that the matrix Y is a unit block triangular matrix).

Thus total count is $\dfrac{16n^3}{3} + \dfrac{k+5r}{2} n^2$ flops.

4.2 Comparison of Efficiency.

Van Dooren's scalar algorithm requires about $(7 + 2r)n^3$ flops. [Note: the flop count given in Van Dooren (1984) is nearly one half of that given here; this is because a "flop" is counted there as a multiplication / division coupled with an addition / subtraction.]

Also, it can be shown that the recent block algorithm of Datta and Sarkissian (2000) requires about $52n^3/3$ flops.

Thus the proposed algorithm is more than three times faster than both Van Dooren's and the Datta-Sarkissian algorithms.

Besides, the proposed algorithm is suitable for implementations using the recently developed and widely-used scientific computing software package LAPACK (Anderson et al.(1995)), since it is composed of BLAS-3 operations such as matrix-matrix multiplications, QR factorizations and solutions of triangular systems with multiple right hand sides.

5 Illustrative numerical example.

Example 5.1 Design of a reduced order observer for the Helicopter problem discussed in Doyle and Stein (1981).

$$A = \begin{bmatrix} -0.0200 & 0.0050 & 2.4000 & -32.0000 \\ -0.1400 & 0.4400 & -1.3000 & -3.0000 \\ 0 & 0.0180 & -1.6000 & 1.2000 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 57.3000 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.14 & -0.12 \\ 0.36 & -8.6 \\ 0.35 & 0.009 \\ 0 & 0 \end{bmatrix}, \mathcal{S} = \{-.1, -.2\}$$

Step 1: The pair (A, C) is reduced to the observable-Hessenberg form $(H, \bar{C})$:

$$P = \begin{bmatrix} -0.9943 & -0.1071 & 0 & 0 \\ 0 & 0 & 0 & -1.0000 \\ 0.1071 & -0.9943 & 0 & 0 \\ 0 & 0 & -1.0000 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} -0.2936 & 2.5407 & -31.9445 & 0.0030 \\ 0.1407 & -1.3264 & -2.2332 & 0.0184 \\ -0.1071 & 0.9943 & 0 & 0 \\ 0 & -1.3075 & -3.0000 & 0.4400 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & -1.0000 \\ -57.3000 & 0 \end{bmatrix}$$

Step 2: $p = \lceil 4/2 \rceil = 2\ ,\ \ q = 1\ ,\ \ \tilde{r} = 2.$

Steps 3 and 4:

$$T = T_{11} = \begin{bmatrix} -0.1000 & 0 \\ 0 & -0.2000 \end{bmatrix}, Y_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 5: skipped $(q = 1)$. **Step 6:** $i = 1$:

Step 7: $j = 1$. Solving the triangular system $Y_{12}H_{21} = -Y_{11}H_{11} + T_{11}Y_{11}$ for Y_{12} gives

$$Y_{12} = \begin{bmatrix} -1.8082 & 0.5681 \\ 1.3139 & 0.1377 \end{bmatrix}$$

Step 8: Solving triangular system $G_1C_1 = Y_{11}H_{12} + Y_{12}H_{22} - T_{11}Y_{12}$ for G_1 , we have

$$G_1 = \begin{bmatrix} -0.3098 & 0.5904 \\ -0.1065 & 0.0416 \end{bmatrix}$$

Step 9 : Forming matrices Y, T and G from the computed blocks:

$$Y = \begin{bmatrix} 1 & 0 & -1.8082 & 0.5681 \\ 0 & 1 & 1.3139 & 0.1377 \end{bmatrix}$$

$$T = \begin{bmatrix} -0.10 & 0 \\ 0 & -0.20 \end{bmatrix}, G = \begin{bmatrix} -0.3098 & 0.5904 \\ -0.1065 & 0.0416 \end{bmatrix}$$

Step 10: Recovering $X = YP^T$:

$$X = \begin{bmatrix} -.9943 & -0.5681 & 0.1071 & 1.8082 \\ -.1071 & -0.1377 & -0.9943 & -1.3139 \end{bmatrix}$$

Verification: $\|XA - TX - GC\|_2 = 3.8154 \times 10^{-16}$ and $\Omega(T) = \{-0.2000, -0.1000\}$. Thus the residue is small and the spectrum of T has been assigned accurately.

Figure 1 shows the relative error between the actual state $x(t)$ and the observed state $\hat{x}(t)$ satisfying (5), for $u(t) = [\delta(t), 0]^T$, with the data above. The system of ordinary differential equations (4) has been solved using **MATLAB** procedure **ode45**.

6 Conclusion

A new block algorithm is proposed for solving the multi-output reduced order Sylvester-observer equation. The algorithm is a generalization of the well-known scalar algorithm by Van Dooren (1984), but uses the block Arnoldi method for reduction to the observer-Hessenberg form. A similar block algorithm was proposed earlier by Shafai and Bhattacharyya (1988) for pole assignment.

The proposed algorithm is numerically more efficient than Van Dooren's and the Datta-Sarkissian algorithms. Furthermore, the algorithm is well-

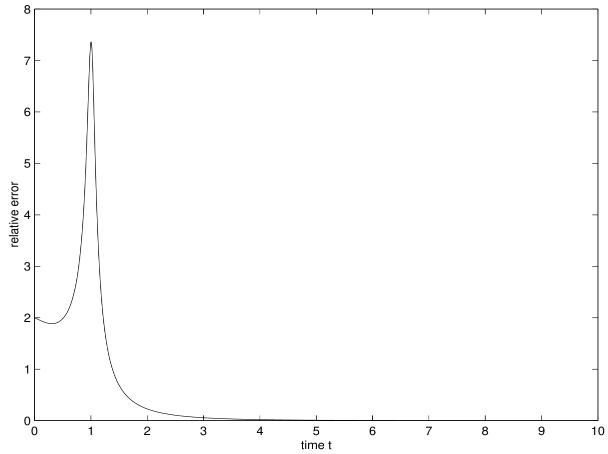

Figure 1: Relative error between the Actual States and the Observed States is converging to zero quite fast as time t increases.

suited for high performance computing using LAPACK An extension of the algorithm to construct a functional observer is being investigated. The possibility of constructing a minimum-norm solution to the Sylvester-observer equation by exploiting the full freedom of choosing the matrix T will also be a topic of future research.

References

[1] E. Anderson et al, LAPACK *User's Guide, Second Edition.* SIAM, Philadelphia, PA, 1995.

[2] S.P. Bhattacharyya and E. DeSouza, *Pole assignment via Sylvester's Equation*, Syst. Contr. Letter., vol 1, pp. 261-283, 1982.

[3] C. Bischof, B. Datta and A. Purkayastha, *A parallel algorithm for the Sylvester observer equation*, SIAM J. Sci. Comp., 17, 686-698, 1996.

[4] D.L. Boley, *Computing the Controllability/Observability Decomposition of a Linear Time-Invariant Dynamic System, A Numerical Approach*, PhD. thesis, Report STAN-CS-81-860, Dept. Comp. Sci., Stanford University, 1981.

[5] D. Calvetti, B.Lewis and L. Reichel,*On the solution of large Sylvester-observer equation*, Num. Lin. Alg. Appl. (to appear in 2001).

[6] J.B.Carvalho and B.N.Datta,*A block algorithm for multi-input eigenvalue assignment problem*, to appear in the proc. IFAC Symposium on Systems, Structure and Control, Prague, 2001.

[7] C.-T.Chen, *Linear System Theory and Design*, Holt, Rinehart and Winston, NY, 1984.

[8] B.N. Datta and Y.Saad, *Arnoldi methods for large Sylvester-like observer matrix equations, and an associated algorithm for partial spectrum assignment*, Linear Algebra Appl. 154/156(1991),pp. 225-244.

[9] B.N.Datta, *Linear and numerical linear algebra in control theory: Some research problems*, Lin. Alg. Appl. 197/198 (1994) , pp. 755-790.

[10] B.N. Datta, *Numerical Linear Algebra and Applications*, Brooks/Cole Publishing Company, Pacific Grove, 1995.

[11] B.N.Datta and C.Hetti,*Generalized Arnoldi methods for the Sylvester- observer equation and the multi-input pole placement problem*, Proc. 36^{th} IEEE Conference on Decision and Control, 1997, pp.4379-4383.

[12] B.N. Datta, *Numerical Methods for Control Systems Design and Analysis*, Academic Press, to appear in 2001.

[13] B.N. Datta and D. Sarkissian, *Block algorithms for state estimation and functional observers*, Proc. IEEE Internation Symposium on Computer-Aided Control System Design, Anchorage, Alaska, Sept. 25-27, 2000, pp. 19-23.

[14] P.Van Dooren,*The generalized eigenstructure problem in linear system theory*, IEEE Trans. Automat. Contr. vol AC-26, 1981, pp. 111-129.

[15] P. Van Dooren, *Reduced order observers: A new algorithm and proof*, Systems Control Lett. 4(1984), pp. 243-251.

[16] J.C. Doyle and G.Stein, *Robustness with observers*, IEEE Trans. Automat. Contr. AC-26,1981, pp.4-16.

[17] G. Golub, C. van Loan,*Matrix Computations*, 3rd. ed., Johns Hopkins Univ. Press, 1996.

[18] G.Golub, S.Nash and C.Van Loan,*A Hessenberg-Schur method for the problem $AX + XB = C$*, IEEE Trans. Automat. Contr. vol AC-24,1979, pp. 909-913.

[19] D. Luenberger,*Observers for multivariable systems*, IEEE Trans. Automat. Contr. vol AC-11, 1966, pp. 190-197.

[20] MathWorks, Inc., The MATLAB User's Guide. South Natick, MA, 1992.

[21] Y. Saad, *Numerical Methods for Large Eigenvalue Problems: Theory and algorithms*, Wiley, NY, 1992.

[22] B. Shafai and S.P. Bhattacharyya, *An algorithm for pole placement in high-order multi-variable systems*, IEEE Trans. Automat. Control, vol. 33, no. 9, pp 870-876, 1988.

[23] E.DeSouza , S.P. Bhattacharyya, *Controllability, observability and the solution of $AX - XB = C$*, Lin. Alg. Appl. 39(1981), pp. 167-168.