WEIGHTED SHADOWING FOR DELAY DIFFERENTIAL EQUATIONS

LUCAS BACKES, DAVOR DRAGIČEVIĆ, MIHÁLY PITUK, AND LOKESH SINGH

Abstract. After introducing the notion of weighted shadowing, we give sufficient conditions under which certain nonlinear perturbations of nonautonomous linear delay differential equations exhibit the shadowing property with respect to a given weight function. As a corollary, it is shown that if the unperturbed equation admits a shifted exponential dichotomy and the Lipschitz constant of the nonlinear term is sufficiently small, then the perturbed system has a shadowing property with respect to an exponential weight function. An application to differential equations with small delay is given. The results are new even in the case of the standard shadowing property.

1. Introduction

It is well-known that, in general, it is either very difficult or even impossible to explicitly solve a given differential equation. In order to overcome this problem, various numerical schemes dealing with numerous classes of differential equations have been proposed. However, any numerical scheme will, in general, produce only an approximate solution of a differential equation. Naturally, any information given by an approximate solution will be useful only in situations when in a vicinity of this approximate solution there exists an exact solution of our differential equation. This observation gives rise to the notion of *shadowing*. Roughly speaking, a given differential equation is said to be *shadowable* if close to its approximate solution we can construct its exact solution. We note that this notion includes the notion of the Hyers-Ulam stability (see [10]) as a particular case. We emphasize that the shadowing theory in the context of smooth dynamical system theory is well-developed (see [25, 26]).

Recently, several authors have investigated the connection between the notions of shadowing and hyperbolicity for difference and differential equations. For results dealing with linear autonomous or periodic dynamics, we refer to [6, 7, 8, 9, 30] and references therein. The general case of nonautonomous and nonlinear dynamics without any periodicity assumptions was treated in [2, 3]. Roughly speaking, the results in [2, 3] assert the following: if the linear part of dynamics exhibits exponential dichotomy (or more generally exponential trichotomy) and the nonlinear part is Lipschitz with a sufficiently small Lipschitz constant, then the nonlinear dynamics exhibits the shadowing property. In [11, 19], analogous results have been obtained for difference equations with infinite delay and partial difference equations, respectively. In [4, 5, 27], the authors obtained similar shadowing type results under much weaker assumptions on the linear part of the dynamics. Finally, for some other relevant results about Hyers-Ulam stability for various equations that rely on different approaches, we refer to [14, 22, 28, 29] and references therein.

The main objective of the present paper is to introduce and to study the so-called weighted shadowing property for semilinear delay differential equations. We first formulate an abstract general result (see Theorem 2.2), which will be applied to the case when the linear part admits a shifted exponential dichotomy (see Theorem 2.3). As a consequence, we will show that sufficiently small perturbations of nonautonomous linear systems with small delays always exhibit an exponentially weighted shadowing property. We remark that the notion of weighted shadowing was motivated by the last property of differential equations with small delays. For these systems standard shadowing in general does not hold, but they possess an exponentially weighted shadowing property. Although the related notion of Hyers–Ulam stability has been investigated for several classes of delay equations (see, e.g., [18, 23]), to the best of our knowledge, Theorems 2.2 and 2.3 are new even for the standard shadowing property in the linear case.

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2. Main results

Given $r > 0$, let $C = C([-r, 0], \mathbb{R}^d)$ denote the Banach space of continuous functions from $[-r, 0]$ into $\mathbb{R}^d$ with the supremum norm $\|\phi\| := \sup_{-r \le \theta \le 0} |\phi(\theta)|$ for $\phi \in C$, where $|\cdot|$ is any norm on $\mathbb{R}^n$.

Consider the nonlinear delay differential equation

$$x'(t) = L(t)x_t + f(t, x_t) \tag{2.1}$$

as a perturbation of the linear nonautonomous equation

$$x'(t) = L(t)x_t, \tag{2.2}$$

where $x_t \in C$ is defined by $x_t(\theta) := x(t + \theta)$ for $\theta \in [-r, 0]$, $L(t)\colon C \to \mathbb{R}^d$ is a bounded linear functional for each $t \in \mathbb{R}$ and the map $\mathbb{R} \times C \ni (t, \phi) \mapsto L(t)\phi \in \mathbb{R}^d$ is continuous. We shall assume that the nonlinearity $f\colon \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous function such that for some Lipschitz constant $c \ge 0$,

$$|f(t, \phi) - f(t, \psi)| \le c\|\phi - \psi\|, \qquad t \in \mathbb{R},\ \phi, \psi \in C. \tag{2.3}$$

Given $\sigma \in \mathbb{R}$, by a *solution of Eq.* (2.1) *on* $[\sigma, \infty)$, we mean a continuous function $x\colon [\sigma - r, \infty) \to \mathbb{R}^d$ which is differentiable on $[\sigma, \infty)$ and satisfies Eq. (2.1) for $t \ge \sigma$. (By the derivative at σ, we mean the right-hand side derivative.) It is well-known [16] that under the above hypotheses, for every $(\sigma, \phi) \in \mathbb{R} \times C$, Eq. (2.1) has a unique solution on $[\sigma, \infty)$ with initial value $x_\sigma = \phi$.

We will consider the following generalization of the classical notion of the Lipschitz shadowing property (Hyers–Ulam stability) for Eq. (2.1) on $[\sigma, \infty)$, where $\sigma \in \mathbb{R}$.

Definition 2.1. Let $w\colon [\sigma, \infty) \to (0, \infty)$ be a positive and continuous *weight function*. Eq. (2.1) is called *w-shadowable on* $[\sigma, \infty)$ if there exists a constant $K > 0$ with the following property: if $\delta > 0$ and $y\colon [\sigma - r, \infty) \to \mathbb{R}^d$ is a continuous function which is continuously differentiable on $[\sigma, \infty)$ and satisfies

$$w(t)|y'(t) - L(t)y_t - f(t, y_t)| \le \delta, \qquad t \ge \sigma, \tag{2.4}$$

then Eq. (2.1) has a solution x on $[\sigma, \infty)$ such that

$$w(t)\|x_t - y_t\| \le K\delta, \qquad t \ge \sigma. \tag{2.5}$$

The function y satisfying (2.4) is called a *w-weighted δ-pseudosolution* of Eq. (2.1) on $[\sigma, \infty)$. If $w \equiv 1$, then the above definition reduces to the standard notion of the Lipschitz shadowing property [26].

Our aim in this paper is to give sufficient conditions under which (2.1) exhibits a weighted shadowing property. We emphasize that our results are new even in the standard case ($w \equiv 1$).

Let $(T(t, s))_{t \ge s}$ denote the solution operator of the linear equation (2.2) on C defined by $T(t, s)\phi = x_t(s, \phi)$ for $t \ge s$ and $\phi \in C$, where $x(s, \phi)$ is the unique solution of Eq. (2.2) on $[s, \infty)$ with initial value $x_s = \phi$. Throughout the paper, we shall assume as a *standing assumption* that there exists a family of bounded projections $(P(t))_{t \in \mathbb{R}}$ on C with the following properties:

(i) the map $\mathbb{R} \ni t \mapsto P(t) \in C$ is continuous,
(ii) $P(t)T(t, s) = T(t, s)P(s)$ whenever $t, s \in \mathbb{R}$ and $t \ge s$,
(iii) the linear operator $T(t, s)|_{\ker P(s)}\colon \ker P(s) \to \ker P(t)$ is onto and invertible whenever $t, s \in \mathbb{R}$ and $t \ge s$.

Define $T(t, s) := \big(T(s, t)|_{\ker P(t)}\big)^{-1}$ for $t \le s$ so that $T(t, s)\colon \ker P(s) \to \ker P(t)$ is a bounded linear operator for $t \le s$ and let $Q(t) := I - P(t)$ for $t \in \mathbb{R}$.

Our main result is the following theorem.

Theorem 2.2. *Let $w\colon [\sigma, \infty) \to (0, \infty)$ be a positive and continuous weight function. Suppose that*

$$\frac{\|T(\sigma, t)Q(t)\|}{w(t)} \longrightarrow 0, \qquad t \to \infty, \tag{2.6}$$

and there exists $M > 0$ such that

$$w(t)\bigg(\int_\sigma^t \frac{\|T(t, s)P(s)\|}{w(s)}\, ds + \int_t^\infty \frac{\|T(t, s)Q(s)\|}{w(s)}\, ds\bigg) \le M, \qquad t \ge \sigma. \tag{2.7}$$

Assume also that

$$c < \frac{1}{M}, \tag{2.8}$$

where c is the Lipschitz constant of f from (2.3). If y is a w-weighted δ-pseudosolution of Eq. (2.1) on $[\sigma,\infty)$ for some $\delta>0$, then there exists a unique solution x of Eq. (2.1) on $[\sigma,\infty)$ such that

$$P(\sigma)x_\sigma = P(\sigma)y_\sigma \qquad \text{and} \qquad w(t)\|x_t-y_t\| \le \frac{\delta M}{1-cM}, \qquad t\ge\sigma. \tag{2.9}$$

In particular, Eq. (2.1) is w-shadowable on $[\sigma,\infty)$.

The proof of Theorem 2.2 will be given in Section 3.

The hypotheses of Theorem 2.2 about the linear part of Eq. (2.1) are satisfied with an appropriate weight function if Eq. (2.2) admits a shifted exponential dichotomy on $\mathbb{R}$ in the sense of the following definition [17]. Let $\gamma\in\mathbb{R}$. We say that the linear equation (2.2) admits a *shifted exponential dichotomy on $\mathbb{R}$ with splitting at γ* if, in addition to the standing assumptions (i)–(iii), there exist constants $D>0$ and $\lambda>0$ such that

$$\|T(t,s)P(s)\| \le De^{(\gamma-\lambda)(t-s)}, \qquad t\ge s, \tag{2.10}$$

$$\|T(t,s)Q(s)\| \le De^{(\gamma+\lambda)(t-s)}, \qquad t\le s. \tag{2.11}$$

A simple calculation shows that under conditions (2.10) and (2.11) both hypotheses (2.6) and (2.7) of Theorem 2.2 are satisfied with $w(t) := \exp_{-\gamma}(t) = e^{-\gamma t}$ and $M := \frac{2D}{\lambda}$ independently of $\sigma\in\mathbb{R}$. Therefore, as a consequense of Theorem 2.2, we have the following result.

Theorem 2.3. *Suppose that Eq. (2.2) admits a shifted exponential dichotomy on $\mathbb{R}$ with splitting at $\gamma\in\mathbb{R}$. If*

$$c < \frac{\lambda}{2D} \tag{2.12}$$

with $D,\lambda>0$ as in (2.10) and (2.11), then, for every $\sigma\in\mathbb{R}$, Eq. (2.1) is $\exp_{-\gamma}$-shadowable on $[\sigma,\infty)$.

Finally, we give an application of Theorem 2.3 to differential equations with small delays [1, 12, 13]. Suppose that the linear functional in Eq. (2.2) is represented by the Riemann–Stieltjes integral

$$L(t)\phi = \int_{-r}^{0} d_\theta[\eta(t,\theta]\phi(\theta), \qquad t\in\mathbb{R},\ \phi\in C,$$

where the kernel $\eta\colon \mathbb{R}\times[-r,0]\to\mathbb{R}^{d\times d}$ is a matrix function such that, for each $t\in\mathbb{R}$, $\eta(t,\cdot)$ is of bounded variation on $[-r,0]$. Assume also that there exists $K>0$ such that $\operatorname{Var}_{[-r,0]}\eta(t,\cdot)\le K$ for $t\in\mathbb{R}$ and

$$Kre<1. \tag{2.13}$$

It is known [1, Theorem 1.1] that in this case there exists a unique *special matrix solution* $X\colon\mathbb{R}\to\mathbb{R}^{d\times d}$ of Eq. (2.2) such that

$$X'(t) = \int_{-r}^{0} d_\theta[\eta(t,\theta]X(t+\theta), \qquad t\in\mathbb{R},$$

$X(0)=I$, $\det X(t)\ne 0$ for $t\in\mathbb{R}$ and $\sup_{t\le 0}\left[|X(t)|e^{t/r}\right]<\infty$, where $|\cdot|$ denotes the matrix norm induced by the given norm $|\cdot|$ on $\mathbb{R}^d$. Moreover, if $(\sigma,\phi)\in\mathbb{R}\times C$, then for the solution x of Eq. (2.2) with initial value $x_\sigma=\phi$ there exists a unique vector $l=l(\sigma,\phi)\in\mathbb{R}^d$ such that

$$\sup_{t\ge\sigma}\left[|x(t)-X(t)l|e^{t/r}\right]<\infty \tag{2.14}$$

(see [1, Theorem 1.3]). In particular, $x(t)-X(t)l\to 0$ as $t\to\infty$. In [1, Theorem 4.1], it is shown that

$$l = l(\sigma,\phi) = \langle Y^\sigma,\phi,\sigma\rangle,$$

where Y is a special matrix solution of the associated formal adjoint equation (see [1, Theorem 3.1]) and the symbol in the angle brackets denotes the analog of Hale's bilinear form (see [1, p. 388]). The proof of [1, Theorem 4.1] implies the existence of a family of bounded projections $(P(t))_{t\in\mathbb{R}}$ on C defined by

$$P(t)\phi := \phi - X_t\langle Y^t,\phi,t\rangle, \qquad t\in\mathbb{R},\ \phi\in C$$

(see [1, p. 403]). Using the fact that the quantity $\langle Y^t, x_t, t\rangle$ is constant along the solutions x of Eq. (2.2) (see [1, Lemma 4.1]), it is easily seen that the projections $P(t)$, $t \in \mathbb{R}$, satisfy the standing assumptions (i)–(iii). In addition, the exponential estimates in [1, p. 403] show that if λ_0 is the unique root of the equation

$$\lambda = Ke^{\lambda r}$$

in $(0, 1/r)$, then, for every $\gamma \in (-1/r, -\lambda_0)$, Eq. (2.2) admits a shifted exponential dichotomy on $\mathbb{R}$ with splitting at γ. Therefore, Theorem 2.3 applies and we conclude that if c satisfies the smallness condition (2.12), then the perturbed equation (2.1) is $\exp_{-\gamma}$-shadowable on every interval $[\sigma, \infty)$, where $\sigma \in \mathbb{R}$.

3. Proof of the main theorem

In this section, we give a proof of Theorem 2.2 which has been inspired by the analytical proofs of the shadowing lemma [20, 24]. As a preparation for the proof, we establish two lemmas.

Lemma 3.1. *Suppose that the hypotheses of Theorem 2.2 are satisfied and let y be a w-weighted δ-pseudosolution of Eq. (2.1) on $[\sigma, \infty)$ for some $\delta > 0$. Then x is a solution of Eq. (2.1) on $[\sigma, \infty)$ satisfying (2.9) if and only if $z = x - y$ is a solution of the equation*

$$z'(t) = L(t)z_t + f(t, y_t + z_t) + L(t)y_t - y'(t) \tag{3.1}$$

on $[\sigma, \infty)$ satisfying

$$P(\sigma)z_\sigma = 0 \qquad \text{and} \qquad w(t)\|z_t\| \le \frac{\delta M}{1 - cM}, \qquad t \ge \sigma. \tag{3.2}$$

Lemma 3.1 can be verified by direct differentiation. Its straightforward proof is omitted.

The following lemma will play a key role in the proof of Theorem 2.2. It shows that the existence and uniqueness of the solution of Eq. (3.1) on $[\sigma, \infty)$ satisfying (3.2) can be reduced to a fixed point problem.

Lemma 3.2. *Suppose that the hypotheses of Theorem 2.2 are satisfied and let y be a w-weighted δ-pseudosolution of Eq. (2.1) on $[\sigma, \infty)$ for some $\delta > 0$. If z is a solution of Eq. (3.1) on $[\sigma, \infty)$ satisfying (3.2), then the continuous function $\zeta \colon [\sigma, \infty) \to C$ defined by the segments $\zeta(t) := z_t$, $t \ge \sigma$, satisfies*

$$\|\zeta\|_w := \sup_{t \ge \sigma} \big[\, w(t)\|\zeta(t)\|\, \big] \le \frac{\delta M}{1 - cM} \tag{3.3}$$

and the following limit relation in C,

$$\zeta(t) = \lim_{n \to \infty} \int_\sigma^t T(t,s)P(s)\Gamma^n h_\zeta(s)\, ds - \lim_{n \to \infty} \int_t^\infty T(t,s)Q(s)\Gamma^n h_\zeta(s)\, ds \tag{3.4}$$

for $t \ge \sigma$, where

$$h_\zeta(t) := f(t, y_t + \zeta(t)) + L(t)y_t - y'(t), \qquad t \ge \sigma, \tag{3.5}$$

and $\Gamma^n \colon \mathbb{R}^d \to C$ is defined by

$$(\Gamma^n x)(\theta) := \begin{cases} (1 + n\theta)x & \text{if } \theta \in [-1/n, 0] \\ 0 & \text{if } \theta \in [-r, -1/n) \end{cases} \tag{3.6}$$

for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, $n > 1/r$.
Conversely, if $\zeta \colon [\sigma, \infty) \to C$ is a continuous function satisfying (3.3) and (3.4), then the function $z \colon [\sigma - r, \infty) \to \mathbb{R}^d$ defined by

$$z(t) := \begin{cases} [\zeta(t)](0) & \text{if } t \ge \sigma \\ [\zeta(\sigma)](t - \sigma) & \text{if } t \in [\sigma - r, \sigma] \end{cases} \tag{3.7}$$

is a solution of Eq. (3.1) on $[\sigma, \infty)$ satisfying (3.2).

The proof of Lemma 3.2 will be based on the following version of the variation of constants formula for the nonhomogeneous equation

$$x'(t) = L(t)x_t + h(t), \tag{3.8}$$

where $L(t)$, $t \in \mathbb{R}$, has the meaning from (2.2) and $h \colon [\sigma, \infty) \to \mathbb{R}^d$ is a continuous function.

Theorem 3.3. *Under the above hypotheses, for every $\phi \in C$, the segments of the (unique) solution x of Eq. (3.8) on $[\sigma, \infty)$ with initial value $x_\sigma = \phi$ satisfy the following limit relation in C,*

$$x_t = T(t,\sigma)\phi + \lim_{n\to\infty} \int_\sigma^t T(t,s)\Gamma^n h(s)\,ds, \qquad t \ge \sigma. \tag{3.9}$$

Theorem 3.3 is a consequence of a more general result for abstract functional differential equations [21, Theorem 4.2].

Proof of Lemma 3.2. Consider the first part of the lemma. Suppose that z is a solution of Eq. (3.1) on $[\sigma, \infty)$ satisfying (3.2). The continuity of the function ζ defined by the segments of z on $[\sigma, \infty)$ is well-known (see [16, Chap. 2, Sec. 2.2, Lemma 2.1]). Clearly, (3.3) is only a reformulation of the second relation in (3.2) and the function h_ζ defined by (3.5) is continuous on $[\sigma, \infty)$. Since $z_t = \zeta(t)$ for $t \ge \sigma$, from (3.1) and (3.5), by the application of Theorem 3.3 with $h = h_\zeta$, we conclude that

$$\zeta(t) = T(t,\sigma)\zeta(\sigma) + \lim_{n\to\infty} \int_\sigma^t T(t,s)\Gamma^n h_\zeta(s)\,ds, \qquad t \ge \sigma.$$

From this, we find that

$$P(t)\zeta(t) = T(t,\sigma)P(\sigma)\zeta(\sigma) + \lim_{n\to\infty} \int_\sigma^t T(t,s)P(s)\Gamma^n h_\zeta(s)\,ds, \qquad t \ge \sigma.$$

By virtue of (3.2), we have that $P(\sigma)\zeta(\sigma) = P(\sigma)z_\sigma = 0$. Therefore,

$$P(t)\zeta(t) = \lim_{n\to\infty} \int_\sigma^t T(t,s)P(s)\Gamma^n h_\zeta(s)\,ds, \qquad t \ge \sigma. \tag{3.10}$$

A similar argument yields

$$Q(t)\zeta(t) = T(t,\sigma)Q(\sigma)\zeta(\sigma) + \lim_{n\to\infty} \int_\sigma^t T(t,s)Q(s)\Gamma^n h_\zeta(s)\,ds, \qquad t \ge \sigma. \tag{3.11}$$

Hence

$$Q(\sigma)\zeta(\sigma) = T(\sigma,t)Q(t)\zeta(t) - \lim_{n\to\infty} \int_\sigma^t T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds, \qquad t \ge \sigma. \tag{3.12}$$

From (3.3), it follows that

$$\|T(\sigma,t)Q(t)\zeta(t)\| \le \frac{\|T(\sigma,t)Q(t)\|}{w(t)} w(t)\|\zeta(t)\| \le \frac{\|T(\sigma,t)Q(t)\|}{w(t)} \frac{\delta M}{1-cM}, \qquad t \ge \sigma.$$

This, together with (2.6), yields

$$T(\sigma,t)Q(t)\zeta(t) \longrightarrow 0, \qquad t \to \infty. \tag{3.13}$$

From (2.3) and (3.5), we obtain for $t \ge \sigma$,

$$\begin{aligned} |h_\zeta(t)| &= |f(t, y_t + \zeta(t)) - f(t, y_t) + f(t, y_t) + L(t)y_t - y'(t)| \\ &\le |f(t, y_t + \zeta(t)) - f(t, y_t)| + |f(t, y_t) + L(t)y_t - y'(t)| \\ &\le c\|\zeta(t)\| + |f(t, y_t) + L(t)y_t - y'(t)|. \end{aligned}$$

From this, (2.4) and (3.3), we find that

$$w(t)|h_\zeta(t)| \le c\frac{\delta M}{1-cM} + \delta = \frac{\delta}{1-cM}, \qquad t \ge \sigma.$$

Taking into account that $\|\Gamma^n x\| = |x|$ for $x \in \mathbb{R}^d$ and $n > 1/r$, the last inequality implies that

$$w(t)\|\Gamma^n h_\zeta(t)\| \le \frac{\delta}{1-cM}, \qquad t \ge \sigma, \tag{3.14}$$

whenever ζ satisfies (3.3). From this and (2.6), we conclude that if $\sigma \le t_1 \le t_2$ and $n > 1/r$, then

$$\begin{aligned}&\left\| \int_\sigma^{t_1} T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds - \int_\sigma^{t_2} T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds \right\| \\ &\le \int_{t_1}^{t_2} \|T(\sigma,s)Q(s)\|\|\Gamma^n h_\zeta(s)\|\,ds \\ &\le \frac{\delta}{1-cM}\int_{t_1}^{\infty} \frac{\|T(\sigma,s)Q(s)\|}{w(s)}\,ds \longrightarrow 0, \qquad t_1\to\infty.\end{aligned}$$

This implies that the limit $\lim_{t\to\infty}\int_\sigma^t T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds$ exists uniformly for $n > 1/r$, while (3.12) shows that the limit $\lim_{n\to\infty}\int_\sigma^t T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds$ exists for each $t \ge \sigma$. From this, it follows by a Moore–Osgood type theorem (see Appendix) that we can interchange the order of the limits

$$\lim_{n\to\infty}\lim_{t\to\infty}\int_\sigma^t T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds = \lim_{t\to\infty}\lim_{n\to\infty}\int_\sigma^t T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds. \tag{3.15}$$

Letting $t\to\infty$ in (3.12) and using (3.13) and (3.15), we find that

$$Q(\sigma)\zeta(\sigma) = -\lim_{n\to\infty}\int_\sigma^\infty T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds, \qquad t\ge\sigma. \tag{3.16}$$

Substituting (3.16) into (3.11), we conclude that

$$Q(t)\zeta(t) = -\lim_{n\to\infty}\int_t^\infty T(t,s)Q(s)\Gamma^n h_\zeta(s)\,ds, \qquad t\ge\sigma. \tag{3.17}$$

Finally, adding Eqs. (3.10) and (3.17), we obtain (3.4).

Now we prove the second part of the lemma. Suppose that $\zeta\colon[\sigma,\infty)\to C$ is a continuous function satisfying (3.3) and (3.4). Define $h := h_\zeta$ with h_ζ as in (3.5) and let v denote the solution of the nonhomogeneous equation (3.8) on $[\sigma,\infty)$ with initial value $v_\sigma = \zeta(\sigma)$. By Theorem 3.3, we have that

$$v_t = T(t,\sigma)\zeta(\sigma) + \lim_{n\to\infty}\int_\sigma^t T(t,s)\Gamma^n h_\zeta(s)\,ds, \qquad t\ge\sigma. \tag{3.18}$$

Applying the projections $P(t)$ and $Q(t)$ to Eq. (3.18), we conclude that both limits $\lim_{n\to\infty}\int_\sigma^t T(t,s)P(s)\Gamma^n h_\zeta(s)\,ds$ and $\lim_{n\to\infty}\int_\sigma^t T(t,s)Q(s)\Gamma^n h_\zeta(s)\,ds$ exist. From this and (3.4) with $t=\sigma$, it follows that the limit relation (3.18) can be written as

$$\begin{aligned} v_t &= T(t,\sigma)\Big(-\lim_{n\to\infty}\int_\sigma^\infty T(\sigma,s)Q(s)\Gamma^n h_\zeta(s)\,ds\Big) + \lim_{n\to\infty}\int_\sigma^t T(t,s)\Gamma^n h_\zeta(s)\,ds \\ &= -\lim_{n\to\infty}\int_\sigma^\infty T(t,s)Q(s)\Gamma^n h_\zeta(s)\,ds + \lim_{n\to\infty}\int_\sigma^t T(t,s)(P(s)+Q(s))\Gamma^n h_\zeta(s)\,ds \\ &= -\lim_{n\to\infty}\int_t^\infty T(t,s)Q(s)\Gamma^n h_\zeta(s)\,ds + \lim_{n\to\infty}\int_\sigma^t T(t,s)P(s)\Gamma^n h_\zeta(s)\,ds = \zeta(t)\end{aligned}$$

for $t\ge\sigma$. From this and (3.5), we find that

$$h_\zeta(t) = f(t,y_t+v_t) + L(t)y_t - y'(t), \qquad t\ge\sigma.$$

Hence

$$v'(t) = L(t)v_t + h_\zeta(t) = L(t)v_t + f(t,y_t+v_t) + L(t)y_t - y'(t), \qquad t\ge\sigma,$$

i.e. v is a solution of Eq. (3.1) on $[\sigma,\infty)$. Since $v_t = \zeta(t)$ for $t\ge\sigma$, it follows that $v(t) = v_t(0) = [\zeta(t)](0) = z(t)$ for $t\ge\sigma$, while $v_\sigma = \zeta(\sigma)$ implies for $t\in[\sigma-r,\sigma]$,

$$v(t) = v_\sigma(t-\sigma) = [\zeta(\sigma)](t-\sigma) = z(t).$$

Thus, the function z defined by (3.7) coincides with the solution v of Eq. (3.1) on $[\sigma - r, \infty)$. Since $z_t = v_t = \zeta(t)$ for $t \geq \sigma$, the second relation in (3.2) follows from (3.3). Finally, from (3.4) with $t = \sigma$, we conclude that

$$P(\sigma)z_\sigma = P(\sigma)\zeta(\sigma) = -\lim_{n\to\infty} P(\sigma)\int_\sigma^\infty Q(\sigma)T(\sigma,s)\varGamma^n h_\zeta(s)\,ds.$$

Since $P(\sigma)Q(\sigma) = 0$, this implies that $P(\sigma)z_\sigma = 0$. □

Based on Lemmas 3.1 and 3.2, we can give a short proof of Theorem 2.2.

Proof of Theorem 2.2. Let W denote the vector space of those functions $\zeta\colon [\sigma,\infty) \to C$ which are continuous and satisfy $\|\zeta\|_w < \infty$, where $\|\zeta\|_w$ is defined by (3.3). Clearly, $\|\cdot\|_w$ is a norm on W and $(W, \|\cdot\|_w)$ is a Banach space. Define

$$B := \left\{ \zeta \in W : \|\zeta\|_w \leq \frac{\delta M}{1 - cM} \right\}$$

so that B is a closed subset of W. For $\zeta \in B$ and $t \geq \sigma$, define

$$(\mathcal{F}\zeta)(t) := \lim_{n\to\infty}\int_\sigma^t T(t,s)P(s)\varGamma^n h_\zeta(s)\,ds - \lim_{n\to\infty}\int_t^\infty T(t,s)Q(s)\varGamma^n h_\zeta(s)\,ds,$$

where h_ζ is defined by (3.5). It follows by similar arguments as in the proof of Lemma 3.2 that both limits on the right-hand side exist. (Note that for this part of the proof of Lemma 3.2 we have not used that the values of ζ are the segments of the solution z. We have merely used that ζ is continuous and satisfies (3.3), which certainly holds for $\zeta \in B$.) Using estimate (3.14) from the proof of Lemma 3.2, we find for $\zeta \in B$ and $t \geq \sigma$,

$$\begin{aligned}\|(\mathcal{F}\zeta)(t)\| &\leq \limsup_{n\to\infty}\int_\sigma^t \|T(t,s)P(s)\|\|\varGamma^n h_\zeta(s)\|\,ds \\ &\quad + \limsup_{n\to\infty}\int_t^\infty \|T(t,s)Q(s)\|\|\varGamma^n h_\zeta(s)\|\,ds \\ &\leq \frac{\delta}{1-cM}\left(\int_\sigma^t \frac{\|T(\sigma,s)P(s)\|}{w(s)}\,ds + \int_t^\infty \frac{\|T(\sigma,s)Q(s)\|}{w(s)}\,ds\right).\end{aligned}$$

This, together with (2.7), implies that

$$\|\mathcal{F}\zeta\|_w = \sup_{t\geq\sigma}\big[\,w(t)\|(\mathcal{F}\zeta)(t)\|\big] \leq \frac{\delta M}{1-cM}, \qquad \zeta \in B.$$

Thus, $\mathcal{F}$ is well-defined and $\mathcal{F}(B) \subset B$.

Let $\zeta_1, \zeta_2 \in B$. By virtue of (2.3), we have for $t \geq \sigma$,

$$|h_{\zeta_1}(t) - h_{\zeta_2}(t)| = |f(t, y_t + \zeta_1(t) - f(t, y_t + \zeta_2(t)| \leq c\|\zeta_1(t) - \zeta_2(t)\|.$$

Hence

$$w(t)\|\varGamma^n(h_{\zeta_1}(t) - h_{\zeta_2}(t))\| \leq c\|\zeta_1 - \zeta_2\|_w$$

for $t \geq \sigma$ and $n > 1/r$. From this, we find for $t \geq \sigma$,

$$\begin{aligned}\|(\mathcal{F}\zeta_1)(t) - (\mathcal{F}\zeta_2)(t)\| &\leq \limsup_{n\to\infty}\int_\sigma^t \|T(t,s)P(s)\|\|\varGamma^n(h_{\zeta_1}(s) - h_{\zeta_2}(s))\|\,ds \\ &\quad + \limsup_{n\to\infty}\int_t^\infty \|T(t,s)Q(s)\|\|\varGamma^n(h_{\zeta_1}(s) - h_{\zeta_2}(s))\|\,ds \\ &\leq c\|\zeta_1 - \zeta_2\|_w\left(\int_\sigma^t \frac{\|T(\sigma,s)P(s)\|}{w(s)}\,ds + \int_t^\infty \frac{\|T(\sigma,s)Q(s)\|}{w(s)}\,ds\right).\end{aligned}$$

This, together with (2.7), implies that

$$\|\mathcal{F}\zeta_1 - \mathcal{F}\zeta_2\|_w \leq cM\|\zeta_1 - \zeta_2\|_w \qquad \text{whenever } \zeta_1, \zeta_2 \in B.$$

By virtue of (2.8), we have that $cM < 1$ and hence $\mathcal{F}\colon B \to B$ is a contraction. By Banach's theorem, $\mathcal{F}$ has a unique fixed point ζ in B. According to Lemma 3.2, the function z defined by (3.7) is the unique solution of

Eq. (3.1) on $[\sigma, \infty)$ satisfying (3.2). From this and Lemma 3.1, we conclude that $x := z + y$ is the unique solution of Eq. (2.1) on $[\sigma, \infty)$ satisfying (2.9). □

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Appendix

For completeness, we prove a variant of the Moore–Osgood theorem which is needed for the proof of Lemma 3.2. For analogous results, see [15, Theorem 2.1.4.1 and Remark 2.1.4.1].

Moore–Osgood type theorem. *Let* $(X, \|\cdot\|_X)$ *be a Banach space and* $\sigma \in \mathbb{R}$. *Suppose that* $f_n \colon [\sigma, \infty) \to X$, $n \in \mathbb{N}$, *is a sequence of functions such that:*

(a) *the limit* $g_n := \lim_{t\to\infty} f_n(t)$ *exists uniformly for* $n \in \mathbb{N}$,
(b) *the limit* $h(t) := \lim_{n\to\infty} f_n(t)$ *exists for all* $t \geq \sigma$.

Then the limit

$$l := \lim_{n\to\infty} g_n \tag{A.1}$$

exists and

$$\lim_{t\to\infty} h(t) = l, \tag{A.2}$$

i.e.

$$\lim_{n\to\infty} \lim_{t\to\infty} f_n(t) = \lim_{t\to\infty} \lim_{n\to\infty} f_n(t). \tag{A.3}$$

Proof. First we prove the existence of the limit in (A.1). Let $\epsilon > 0$. Condition (a) guarantees that if we choose t large enough, then

$$\|f_n(t) - g_n\|_X < \frac{\epsilon}{4} \qquad \text{for all } n \in \mathbb{N}. \tag{A.4}$$

Let such t be fixed. Condition (b) implies the existence of n_0 such that

$$\|f_n(t) - h(t)\|_X < \frac{\epsilon}{4} \qquad \text{for } n \geq n_0. \tag{A.5}$$

If $n, m \geq n_0$, then (A.4) and (A.5), together with the triangle inequality, imply that

$$\begin{aligned}\|g_n - g_m\|_X &\leq \|g_n - f_n(t)\|_X + \|f_n(t) - h(t)\|_X \\ &\quad + \|h(t) - f_m(t)\|_X + \|f_m(t) - g_m\|_X < 4\frac{\epsilon}{4} = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, this shows that $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in X. Therefore, the limit in (A.1) exists.

Now we prove the limit relation (A.2). Let $\epsilon > 0$. Condition (a) implies the existence of $\tau \geq \sigma$ such that

$$\|f_n(t) - g_n\|_X < \frac{\epsilon}{3} \qquad \text{whenever } t \geq \tau \text{ and } n \in \mathbb{N}. \tag{A.6}$$

Let $t \geq \tau$. Condition (b) guarantees the existence of n_1 such that

$$\|f_n(t) - h(t)\|_X < \frac{\epsilon}{3} \qquad \text{for } n \geq n_1, \tag{A.7}$$

while (A.1) implies that there exists n_2 such that

$$\|g_n - l\|_X < \frac{\epsilon}{3} \qquad \text{for } n \geq n_2. \tag{A.8}$$

Define $n := \max\{n_1, n_2\}$. Then (A.6), (A.7) and (A.8), together with the triangle inequality, imply that

$$\|h(t) - l\|_X \leq \|h(t) - f_n(t)\|_X + \|f_n(t) - g_n\|_X + \|g_n - l\|_X < 3\frac{\epsilon}{3} = \epsilon.$$

Thus, $\|h(t) - l\|_X < \epsilon$ for all $t \geq \tau$. Since $\epsilon > 0$ was arbitrary, this proves (A.2). ☐

Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, CEP 91509-900, Porto Alegre, RS, Brazil.
E-mail address: lucas.backes@ufrgs.br

Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: ddragicevic@math.uniri.hr

Department of Mathematics, University of Pannonia, Egyetem út 10, 8200 Veszprém, Hungary
E-mail address: pitukm@almos.uni-pannon.hu

Department of Mathematics, University of Rijeka, Radmile Matejčić 2, 51000 Rijeka, Croatia
E-mail address: lokesh.singh@math.uniri.hr