# A ROLEWICZ-TYPE CHARACTERIZATION OF NONUNIFORM BEHAVIOUR

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Abstract. We present necessary and sufficient conditions in the spirit of Rolewicz under which all Lyapunov exponents of a given linear cocycle are either positive or negative. As a consequence, we formulate new conditions for the existence of the so-called tempered exponential dichotomies. We consider cocycles over both maps and flows.

## 1. Introduction

In [11], Datko proved his famous theorem that asserts that for a $C_0$-semigroup $T(t)$, $t \ge 0$ of bounded operators on an arbitrary Hilbert space $X$, the following statements are equivalent:

- $T(t)$ is exponentially stable, i.e. there exist $D, \lambda > 0$ such that
$$\|T(t)\| \le De^{-\lambda t} \quad \text{for } t \ge 0;$$
- for each $x \in X$,
$$\int_0^\infty \|T(t)x\|^2 \, dt < \infty. \tag{1}$$

Since then, the above result has inspired numerous extensions and generalizations for different classes of dynamics. Among many developments, we mention that Pazy [26] showed that the conclusion of Datko's theorem remains valid if (1) is replaced by the requirement that there exists $p > 0$ such that
$$\int_0^\infty \|T(t)x\|^p \, dt < \infty, \quad \text{for each } x \in X.$$
Similar results for discrete semigroups of linear operators were established by Zabczyk [46]. Furthermore, Datko was able to establish the version of his theorem which deals with a nonautonomous dynamics. More precisely, in [12] he proved that for an evolution family $T(t,s)$ on $\mathbb{R}_+ = [0,\infty)$ of bounded operators on an arbitrary Banach space $X$, the following properties are equivalent:

- $T(t,s)$ is (uniformly) exponentially stable, i.e. there exist $D, \lambda > 0$ such that
$$\|T(t,s)\| \le De^{-\lambda(t-s)}, \quad \text{for } t \ge s \ge 0;$$

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- there exists $p > 0$ such that

$$\sup_{s\geq 0}\int_s^\infty \|T(t,s)x\|^p\, dt < \infty, \quad \text{for every } x\in X. \tag{2}$$

A major contribution to this line of the research is due to Rolewicz [34] who proved that in the above characterization of exponential stability for evolution families, condition (2) can be replaced with the requirement that for each $x \in X$ there exists $\alpha(x) > 0$ such that

$$\sup_{s\geq 0}\int_s^\infty N(\alpha(x), \|T(t,s)x\|)\, dt < \infty, \tag{3}$$

where $N\colon (0,\infty)\times[0,\infty)\to[0,\infty)$ is an arbitrary map that satisfies the following properties:

- $N(\cdot, u)$ is non-decreasing for each $u \geq 0$;
- $N(\alpha, \cdot)$ is continuous and non-decreasing for each $\alpha > 0$;
- $N(\alpha, 0) = 0$ and $N(\alpha, u) > 0$ for every $\alpha > 0$ and $u \geq 0$.

We emphasize that (3) includes (2) as a very particular case that corresponds to $N(\alpha, u) = u^p$. We also refer to [33] for similar results in the case of discrete evolution families and to [41] for the simplest known proof of the Rolewicz's result. More recently, many authors have obtained characterizations of exponential stability, instability and dichotomy (including their nonuniform and stochastic versions) for continuous and discrete evolution families (see [6, 8, 14, 18, 19, 21, 24, 30]) in the spirit of Datko, Pazy and Rolewicz. In particular, we recommend [36] for an excellent survey devoted to those developments.

Finally and most importantly for our work, we mention important papers [22, 29, 35, 37, 38] in which the authors obtained characterizations of uniform exponential behaviour for linear cocycles over maps and flows of Datko-Pazy and Rolewicz type (see also [13]). The importance of those results stems from the fact that the notion of a linear cocycle arises naturally in the study of the nonautonomous dynamical systems. Indeed, the smooth ergodic theory builds around the study of the derivative cocycle associated either to a map or a flow (see Chapters 5 and 6 in [4]). Moreover, cocycles describe solutions of variational equations and Cauchy problems with unbounded coefficients (see Chapter 6 in [10]). Finally, we note that cocycles describe solutions of stochastic differential equations (see [1] for details).

The objective of our paper is to formulate Rolewicz-type conditions that imply *nonuniform* exponential stability, instability and dichotomy (via negativity, positivity and nonvanishing of Lyapunov exponents) for linear cocycles over both maps and flows. Our approach builds on that in [15], where the second author has formulated Datko-Pazy conditions that imply nonuniform exponential stability (see [31, 32] for similar results but where conditions are formulated in terms of the so-called Lyapunov norms and hence hard to verify in practice). We emphasize that the results of the present paper represent a nontrivial extension of those in [15]. Moreover, as in [15] we also formulate new conditions for uniform exponential stability, instability and dichotomy of linear cocycles. This is achieved by combining our results with results devoted to the so-called subadditive ergodic optimization obtained by Morris [23] (building on the earlier work of Cao [9], Schreiber [39] and Sturman and Stark [42]).

The relevance of our results stems from the importance of the theory of *nonuniform hyperbolicity* that goes back to the landmark work of Oseledets [25] and particulary Pesin [27, 28] (see [4] for a detailed exposition). This theory represents a nontrivial and far-reaching extension of the theory of uniformly hyperbolic dynamical systems which was initiated by Smale [40]. Hence, it is important to develop tools for detecting nonuniform hyperbolicity and precisely this is a main objective and contribution of the present paper.

The paper is organized as follows. In Section 2.2 we describe necessary and sufficient conditions under which all Lyapunov exponents of a given cocycle are negative and present many consequences of it while in Section 2.3 the case of positive Lyapunov exponents is considered. In Section 2.4 we combine the previous results in order to get the characterization of tempered exponential dichotomies. Section 3 is devoted to extend all the previous results for coycles over continuous-time dynamical systems.

## 2. Cocycles over maps

2.1. **Preliminaries.** Let $M$ be an arbitrary compact metric space and assume that $f\colon M \to M$ is a continuous map. Furthermore, let $X = (X, \|\cdot\|)$ be an arbitrary separable Banach space and let $B(X)$ denote the space of all bounded linear operators on $X$. Finally, set $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. A map $\mathcal{A}\colon M \times \mathbb{N}_0 \to B(X)$ is said to be a *cocycle* over $f$ if:

(1) $\mathcal{A}(q, 0) = \text{Id}$ for each $q \in M$;
(2) $\mathcal{A}(q, n + m) = \mathcal{A}(f^m(q), n)\mathcal{A}(q, m)$ for each $q \in M$ and $n, m \in \mathbb{N}_0$;
(3) the map $A\colon M \to B(X)$ given by
$$A(q) = \mathcal{A}(q, 1), \quad q \in M \tag{4}$$
is strongly continuous, i.e. the map $q \mapsto A(q)x$ is continuous for each $x \in X$.

We recall that the map $A$ given by (4) is called the *generator* of a cocycle $\mathcal{A}$. Let $\mathcal{E}(f)$ denote the set of all ergodic, $f$-invariant Borel probability measures on $M$. Since $M$ is compact and $f$ continuous, we have that $\mathcal{E}(f) \neq \emptyset$. Observe that:

- the map $q \mapsto \|\mathcal{A}(q, n)\|$ is Borel-measurable for each $n \in \mathbb{N}$ (see [17, Lemma 2.4.]);
- it follows from the strong continuity of $A$, compactness of $M$ and the uniform boundness principle that
$$\sup_{q \in M} \|A(q)\| < \infty. \tag{5}$$

Hence, the Kingman's subadditive ergodic theorem [20] implies that for each $\mu \in \mathcal{E}(f)$, there exists $\lambda_\mu(\mathcal{A}) \in [-\infty, \infty)$ such that
$$\lambda_\mu(\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q, n)\|, \quad \text{for } \mu\text{-a.e. } q \in M. \tag{6}$$

The number $\lambda_\mu(\mathcal{A})$ is called the *largest Lyapunov exponent* of a cocycle $\mathcal{A}$ with respect to $\mu$.

We also introduce a class of maps which will play a central role in our results. For a Borel-measurable set $E \subset M$, let $\mathcal{F}(E)$ denote the collection of all maps $\mathcal{N} \colon E \times [0, \infty) \to [0, \infty)$ with the following properties:

(1) $\mathcal{N}(q, 0) = 0$ and $\mathcal{N}(q, t) > 0$ for $q \in E$ and $t > 0$;
(2) $\mathcal{N}(q, \cdot)$ is nondecreasing for $q \in E$;
(3) $\mathcal{N}(\cdot, t)$ is measurable for each $t > 0$.

2.2. **Nonuniform exponential stability.** We start with a result describing necessary and sufficient conditions for a cocycle to be non-uniformly exponentially stable (see also Theorem 2 below).

**Theorem 1.** *For any $\mu \in \mathcal{E}(f)$, the following properties are equivalent:*

(1) $\lambda_\mu(\mathcal{A}) < 0$*;*
(2) *there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) = 1$, a Borel-measurable function $C \colon E \to (0, \infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that*
$$\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q, n)x\|) \le C(q)\mathcal{N}(q, \|x\|) \tag{7}$$
*for $q \in E$ and $x \in X$;*
(3) *there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) > 0$, a Borel-measurable function $C \colon E \to (0, \infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that (7) holds for $q \in E$ and $x \in X$.*

*Proof.* Let us show that (1) implies (2). Assume that $\lambda_\mu(\mathcal{A}) < 0$ and take an arbitrary $\varepsilon > 0$ such that $\lambda_\mu(\mathcal{A}) + \varepsilon < 0$. It follows from (6) that there exists $E \subset M$ satisfying $\mu(E) = 1$ and such that
$$\overline{C}(q) := \sup\{\|\mathcal{A}(q, n)\| e^{-n(\lambda_\mu(\mathcal{A})+\varepsilon)} : n \in \mathbb{N}_0\} < \infty, \tag{8}$$
for $q \in E$. Obviously, $\overline{C}$ is measurable and
$$\|\mathcal{A}(q, n)\| \le \overline{C}(q) e^{n(\lambda_\mu(\mathcal{A})+\varepsilon)} \quad \text{for } q \in E \text{ and } n \in \mathbb{N}_0. \tag{9}$$
Take now any $p > 0$ and define $\mathcal{N} \colon E \times [0, \infty) \to [0, \infty)$ by $\mathcal{N}(q, t) = t^p$. Clearly, $\mathcal{N} \in \mathcal{F}(E)$. It follows from (9) that
$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q, n)x\|) &= \sum_{n=0}^{\infty} \|\mathcal{A}(q, n)x\|^p \\ &\le \overline{C}(q)^p \|x\|^p \sum_{n=0}^{\infty} e^{np(\lambda_\mu(\mathcal{A})+\varepsilon)} \\ &= \frac{\overline{C}(q)^p}{1 - e^{p(\lambda_\mu(\mathcal{A})+\varepsilon)}} \mathcal{N}(q, \|x\|),\end{aligned}$$
and consequently (7) holds with
$$C(q) := \frac{\overline{C}(q)^p}{1 - e^{p(\lambda_\mu(\mathcal{A})+\varepsilon)}} \in (0, \infty).$$
Since (2) trivially implies (3), it remains to prove that (3) implies (1). Therefore, let us assume that there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) > 0$, a Borel-measurable function $C \colon E \to (0, \infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such

that (7) holds for $q \in E$ and $x \in X$. Using Lusin's theorem, we can without any loss of assumption assume that $E$ is compact and that $C$, $\mathcal{N}(\cdot, \frac{1}{2})$ and $\mathcal{N}(\cdot, 1)$ are continuous on $E$. Indeed, otherwise we can simply replace $E$ with its subset of positive measure that has desired properties. For $q \in E$ and $x \in X$, let

$$\|x\|_q := \sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)x\|).$$

It follows from (7) that

$$\mathcal{N}(q, \|x\|) \le \|x\|_q \le C(q)\mathcal{N}(q, \|x\|) \quad \text{for } q \in E \text{ and } x \in X. \tag{10}$$

Since $E$ is compact and $C$ continuous on $E$, we have that there exists $K > 1$ such that

$$\sup_{q\in E} C(q) = \max_{q\in E} C(q) \le K. \tag{11}$$

Set

$$\gamma := 1 - \frac{1}{K} \in (0,1).$$

**Lemma 1.** *For any $m \in \mathbb{N}$, $q \in E$ and $x \in X$, we have that*

$$\|\mathcal{A}(q,m)x\|_{f^m(q)} \le \gamma \|x\|_q. \tag{12}$$

*Proof of the lemma.* Note that

$$\begin{aligned}
\|\mathcal{A}(q,m)x\|_{f^m(q)} &= \sum_{n=0}^{\infty} \mathcal{N}(f^n(f^m(q)), \|\mathcal{A}(f^m(q),n)\mathcal{A}(q,m)x\|) \\
&= \sum_{n=0}^{\infty} \mathcal{N}(f^{n+m}(q), \|\mathcal{A}(q,m+n)x\|) \\
&= \sum_{n=m}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)x\|) \\
&\le \sum_{n=1}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)x\|) \\
&= \|x\|_q - \mathcal{N}(q, \|x\|).
\end{aligned} \tag{13}$$

On the other hand, (10) and (11) imply that

$$\|x\|_q \le K\mathcal{N}(q, \|x\|),$$

and thus

$$\|x\|_q - \mathcal{N}(q, \|x\|) \le (1 - 1/K)\|x\|_q. \tag{14}$$

Combining (13) and (14), we conclude that (12) holds. □

On the other hand, it follows from Poincaré recurrence theorem (see [44, 45]) that $\mu(E') = \mu(E)$, where

$$E' := \{q \in E : f^n(q) \in E \text{ for infinitely many } n \in \mathbb{N}\}.$$

For each $q \in E'$, set

$$\tau(q) := \min\{n \in \mathbb{N} : f^n(q) \in E\} \quad \text{and} \quad \bar{f}(q) := f^{\tau(q)}(q).$$

Moreover, let

$$\overline{A}(q) := \mathcal{A}(q, \tau(q)), \quad q \in E'$$

and consider the cocycle $\overline{\mathcal{A}}$ over $\bar{f}$ and with generator $\overline{A}$. Note that

$$\overline{f}^n(q) = f^{\tau_n(q)}(q) \quad \text{and} \quad \overline{\mathcal{A}}(q,n) = \mathcal{A}(q,\tau_n(q)) \quad \text{for } q \in E' \text{ and } n \in \mathbb{N}, \tag{15}$$

where

$$\tau_n(q) := \sum_{i=0}^{n-1} \tau(\bar{f}^i(q)).$$

**Lemma 2.** *For each $q \in E'$ and $n \in \mathbb{N}$, we have that*

$$\|\overline{\mathcal{A}}(q,n)x\|_{\overline{f}^n(q)} \le \gamma^n \|x\|_q \quad \text{for every } x \in X. \tag{16}$$

*Proof of the lemma.* By (12) and (15), we have that

$$\begin{aligned}
\|\overline{\mathcal{A}}(q,n)x\|_{\overline{f}^n(q)} &= \|\mathcal{A}(q,\tau_n(q))x\|_{f^{\tau_n(q)}(q)} \\
&= \|\mathcal{A}(f^{\tau_{n-1}(q)}(q), \tau_n(q)-\tau_{n-1}(q))\mathcal{A}(q,\tau_{n-1}(q))x\|_{f^{\tau_n(q)}(q)} \\
&\le \gamma \|\mathcal{A}(q,\tau_{n-1}(q))x\|_{f^{\tau_{n-1}(q)}(q)}.
\end{aligned}$$

Now (16) follows by iterating. □

It follows from (10), (11) and (16) that

$$\mathcal{N}(\overline{f}^n(q), \|\overline{\mathcal{A}}(q,n)x\|) \le \|\overline{\mathcal{A}}(q,n)x\|_{\overline{f}^n(q)} \le \gamma^n\|x\|_q \le K\gamma^n \mathcal{N}(q,\|x\|),$$

and thus

$$\mathcal{N}(\overline{f}^n(q), \|\overline{\mathcal{A}}(q,n)x\|) \le K\gamma^n \mathcal{N}(q,1), \tag{17}$$

for every $n \in \mathbb{N}$, $q \in E'$ and $x \in X$ such that $\|x\| \le 1$. Since $\mathcal{N}(\cdot, \frac{1}{2})$ and $\mathcal{N}(\cdot,1)$ are continuous and positive on $E$, there exists $n_0 \in \mathbb{N}$ such that

$$K\gamma^{n_0} \max_{q\in E} \mathcal{N}(q,1) \le \min_{q\in E} \mathcal{N}(q, \frac{1}{2}). \tag{18}$$

By (17) and (18),

$$\mathcal{N}(\overline{f}^{n_0}(q), \|\overline{\mathcal{A}}(q,n_0)x\|) \le \mathcal{N}(\overline{f}^{n_0}(q), \frac{1}{2}),$$

for every $q \in E'$ and $x \in X$ such that $\|x\| \le 1$. Therefore, recalling that $\mathcal{N}(q,\cdot)$ is nondecreasing for $q \in E$,

$$\|\mathcal{A}(q,\tau_{n_0}(q))\| = \|\overline{\mathcal{A}}(q,n_0)\| \le \frac{1}{2} \quad \text{for } q \in E'. \tag{19}$$

Since $\tau_{kn_0}(q) = \sum_{j=0}^{k-1} \tau_{n_0}(\overline{f}^j(q))$, it follows from (19) that

$$\|\mathcal{A}(q,\tau_{kn_0}(q))\| \le \prod_{j=0}^{k-1} \|\mathcal{A}(\overline{f}^j(q), \tau_{n_0}(\overline{f}^j(q)))\| \le \frac{1}{2^k},$$

for $q \in E'$ and $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}\lambda_\mu(\mathcal{A}) &= \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)\| \\ &= \lim_{n\to\infty} \frac{1}{\tau_n(q)} \log\|\mathcal{A}(q,\tau_n(q))\| \\ &= \lim_{k\to\infty} \frac{1}{\tau_{kn_0}(q)} \log\|\mathcal{A}(q,\tau_{kn_0}(q))\| \\ &\le (\log \frac{1}{2}) \cdot \lim_{k\to\infty} \frac{k}{\tau_{kn_0}(q)} \\ &< 0,\end{aligned}$$

where in the last step we use Kac's lemma (see [44]) which implies that

$$\lim_{n\to\infty} \frac{\tau_n(q)}{n} = \frac{1}{\mu(E)} \quad \text{for } \mu\text{-a.e. } q \in E'$$

and thus

$$\lim_{k\to\infty} \frac{k}{\tau_{kn_0}(q)} = \frac{\mu(E)}{n_0} > 0 \quad \text{for } \mu\text{-a.e. } q \in E'.$$

The proof of the theorem is completed. □

**Remark 1.** *We would like to emphasize that Theorem 1 includes* [15, Theorem 1] *as a particular case that corresponds to $\mathcal{N}$ given by $\mathcal{N}(q,t) = t^p$, where $p > 0$.*

**Remark 2.** *Take any $\delta' \ge \delta > 0$. A careful inspection of the proof of Theorem 1 shows that condition (7) can be replaced with the requirement that*

$$\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)x\|) \le C(q)\mathcal{N}(q,\delta'),$$

*for $q \in E$ and $x \in X$ such that $\|x\| = \delta$. Indeed, in this case we have that*

$$\mathcal{N}(q,\|x\|) \le \|x\|_q \le C(q)\mathcal{N}(q,\delta') \quad \text{for } q \in E \text{ and } \|x\| = \delta.$$

*Similarly to the proof of Theorem 1, we can assume that $E$ is a compact set of positive measure such that $\mathcal{N}(\cdot,\delta)$, $\mathcal{N}(\cdot,\delta/2)$ and $\mathcal{N}(\cdot,\delta')$ are continuous on $E$. Furthermore, set*

$$a := \min_{q\in E} \frac{\mathcal{N}(q,\delta)}{\mathcal{N}(q,\delta')} > 0.$$

*By repeating the arguments in the proof of Lemma 1, one can show that (12) holds for $q \in E$ and $x \in X$ with $\|x\| = \delta$, where $\gamma = 1 - a/K$ and $K$ is given by (11). By increasing $K$ if necessary, we can assume that $\gamma \in (0,1)$. Therefore, one can also establish that the conclusion of Lemma 2 holds for each $q \in E'$ and $x \in X$ satisfying $\|x\| = \delta$. Consequently, we have that*

$$\mathcal{N}(\overline{f}^n(q), \|\overline{\mathcal{A}}(q,n)x\|) \le K\gamma^n \mathcal{N}(q,\delta'),$$

*when $q \in E'$ and $\|x\| = \delta$. Choosing $n_0 \in \mathbb{N}$ such that*

$$K\gamma^{n_0} \max_{q\in E} \mathcal{N}(q,\delta') \le \min_{q\in E} \mathcal{N}(q,\delta/2),$$

*we get that (19) holds. Now one can proceed as in the proof of Theorem 1 and establish the negativity of the largest Lyapunov exponent.*

We recall the following result obtained in [15, Theorem 2.] that shows that the cocycle with the negative largest Lyapunov exponent is nonuniformly exponentially stable in the sense of Pesin [4, 27, 28]. Hence, Theorem 1 gives conditions under which the cocycle exhibits nonuniformly stable behaviour.

**Theorem 2.** *Assume that that $\lambda_\mu(\mathcal{A}) < 0$ for some $\mu \in \mathcal{E}(f)$. Then, for each $\varepsilon > 0$ there exists a measurable function $T\colon M \to (0,\infty)$ such that:*

(1) *for $\mu$-a.e. $q \in M$ and $n \in \mathbb{N}_0$,*
$$\|\mathcal{A}(q,n)\| \le T(q)e^{(\lambda_\mu(\mathcal{A})+\varepsilon)n};$$
(2) *for $\mu$-a.e. $q \in M$ and $n \in \mathbb{N}_0$,*
$$T(f^n(q)) \le T(q)e^{\varepsilon n}.$$

We are now in position to formulate new conditions for *uniform* exponential stability of continuous cocycles, i.e. of cocycles with the property that $A\colon M \to B(X)$ is a continuous map. We will say that a Borel subset $E \subset M$ has full-measure if $\mu(E) = 1$ for every $\mu \in \mathcal{E}(f)$. The proof of the following result is analogous to the proof of [15, Theorem 3] but we include it for the sake of completeness.

**Theorem 3.** *Assume that $\mathcal{A}$ is a cocycle over $f$ such that the map $A$ given by (4) is continuous. Then, the following properties are equivalent:*

(1) *there exist a full-measure set $E \subset M$, a Borel-measurable map $C\colon E \to (0,\infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that (7) holds for each $q \in E$ and $x \in X$;*
(2) *$\mathcal{A}$ is uniformly exponentially stable, i.e. there exist $D, \lambda > 0$ such that*
$$\|\mathcal{A}(q,n)\| \le De^{-\lambda n} \quad \text{for every } q \in M \text{ and } n \in \mathbb{N}_0. \tag{20}$$

*Proof.* Proceeding as in the proof of Theorem 1, it is easy to verify that (20) implies that (7) holds for any $q \in M$, $x \in X$ and with a $\mathcal{N}$ given by $N(q,t) = t^p$, for any $p > 0$.

Let us establish the converse. Assume that there exist a full-measure set $E \subset M$, a Borel-measurable map $C\colon E \to (0,\infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that (7) holds for each $q \in E$ and $x \in X$. It follows from Theorem 1 that
$$\lambda_\mu(\mathcal{A}) < 0 \quad \text{for every } \mu \in \mathcal{E}(f). \tag{21}$$
Consider a sequence of maps $(F_n)_{n\in\mathbb{N}}$, where $F_n\colon M \to \mathbb{R}\cup\{-\infty\}$ is given by
$$F_n(q) := \log\|\mathcal{A}(q,n)\| \quad \text{for } q \in M \text{ and } n \in \mathbb{N}, \tag{22}$$
with the convention that $\log 0 := -\infty$. Note that since $A$ and $f$ are continuous, $F_n$ is upper semi-continuous for each $n \in \mathbb{N}$. Moreover,
$$F_{n+m}(q) \le F_n(f^m(q)) + F_m(q), \quad \text{for } m, n \in \mathbb{N} \text{ and } q \in M.$$
Therefore, it follows from [23, Theorem A.3] that there exists $\nu \in \mathcal{E}(f)$ such that
$$\lim_{n\to\infty} \frac{1}{n} \max_{q\in M} F_n(q) = \lambda_\nu(\mathcal{A}),$$
which together with (21) implies that
$$\lim_{n\to\infty} \frac{1}{n} \max_{q\in M} F_n(q) < 0.$$

Hence, there exist $\lambda > 0$ and $n_0 \in \mathbb{N}$ such that

$$\max_{q \in M} F_n(q) \le -\lambda n \quad \text{for every } n \ge n_0,$$

which in a view of (22) readily implies (20). □

Our Theorem 3 is quite similar to the following result established by Sasu and Sasu [35, Theorem 3.1.].

**Theorem 4.** *Assume that $\mathcal{A}$ is a cocycle over $f$. Then, the following properties are equivalent:*

(1) *there exist a nondecreasing function $N\colon [0, \infty) \to [0, \infty)$ satisfying $N(0) = 0$ and $N(t) > 0$ for $t > 0$ and $K, \delta > 0$ such that*

$$\sum_{n=0}^{\infty} N(\|\mathcal{A}(q, n)x\|) \le K, \tag{23}$$

*for $q \in M$ and $x \in X$ such that $\|x\| \le \delta$;*

(2) *there exist $D, \lambda > 0$ such that (20) holds.*

**Remark 3.** *Let us now compare Theorems 3 and 4. We note that in Theorem 4, it is required that (23) holds for each $q \in M$. On the other hand, in the statement of Theorem 3 we require that (7) holds for each $q \in E$, where $E$ is a set of full-measure. At the first glance, it might seem that the latter is an artifical and only minor relaxation of the requirement in the statement of Theorem 4. However, it turns out that quite the opposite is true. More precisely, it follows from the results of Barreira and Schmeling [5] that in many generic situations, $M \setminus E$ can have a complicated structure when $E$ is a set of full-measure. More precisely, it can happen that $M \setminus E$ has the same Hausdorff dimension as $M$ which in particular implies that it is uncountable (and thus nonempty). Moreover, our condition (7) is stated in terms of a map $\mathcal{N}$ which is a function of two variables (just like in the work of Rolewicz [34]), while (23) is expressed in terms of the existence of a map $N$ which is a function of one real variable. The drawback of our Theorem 3 when compared with Theorem 4 is that it requires for the cocycle $\mathcal{A}$ to be (operator norm) continuous. Certainly, this is considerably stronger than to require that the cocycle is only strongly continuous. However, it still includes many interesting classes of dynamics (see [4, 7, 10]). We conclude that Theorem 3 represents a nontrivial extension of Theorem 4 (for continuous cocycles).*

Assume that $\lambda_\mu(\mathcal{A}) \ge 0$. Then, it follows readily from Theorem 1 that there exists $E \subset M$ satisfying $\mu(E) = 1$ and such that for each $q \in E$ there exists $x \in X$ such that

$$\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q, n)x\|) = +\infty.$$

We will now study topological properties of the set of all $x \in X$ for which the above equality holds (for a fixed point $q \in E$).

We begin by introducing some auxiliary notation. Let $[0, \infty)^{\mathbb{N}_0}$ denote the set of all sequences $(a_n)_{n \ge 0}$ with $a_n \in [0, \infty)$ for every $n \ge 0$. Furthermore, let $E \subset M$ be any Borel-measurable set and consider a family $\mathcal{F}'(E)$ that consists of all $\mathcal{N} \in \mathcal{F}(E)$ such that:

(1) $\mathcal{N}\colon M\times[0,\infty)\to[0,\infty)$ is a measurable map;
(2) $\mathcal{N}(q,\cdot)$ is lower semicontinuous for each $q\in E$;
(3) For every $q\in E$, the set
$$\left\{(a_n)_{n\geq 0}\in[0,\infty)^{\mathbb{N}_0}:\sum_{n=0}^\infty \mathcal{N}(f^n(q),a_n)<\infty\right\}$$
is a convex cone.

**Theorem 5.** *Assume that $\lambda_\mu(\mathcal{A})\geq 0$ and take an arbitrary $\mathcal{N}\in\mathcal{F}'(M)$. Then, there exists a Borel-measurable set $E\subset M$ satisfying $\mu(E)=1$ and such that*
$$X_q:=\left\{x\in X:\sum_{n=0}^\infty \mathcal{N}(f^n(q),\|\mathcal{A}(q,n)x\|)=+\infty\right\} \tag{24}$$
*is a residual set, for each $q\in E$.*

*Proof.* It follows from Theorem 1 that there exists a Borel-measurable set $E\subset M$ satisfying $\mu(E)=1$ and such that for every $q\in E$ there exists $x_q\in X$ so that
$$\sum_{n=0}^\infty \mathcal{N}(f^n(q),\|\mathcal{A}(q,n)x_q\|)=+\infty. \tag{25}$$
We claim that $X_q$ is residual for each $q\in E$. We first observe that
$$X_q=\bigcap_{k=1}^\infty X_{q,k},$$
where
$$X_{q,k}:=\left\{x\in X:\sum_{n=0}^\infty \mathcal{N}(f^n(q),\|\mathcal{A}(q,n)x\|)>k\right\}.$$
Take $x\in X_{q,k}$. Hence, there exists $N\in\mathbb{N}$ such that
$$\sum_{n=0}^N \mathcal{N}(f^n(q),\|\mathcal{A}(q,n)x\|)>k.$$
By using lower-semicontinuity of $\mathcal{N}$ and strong continuity of $\mathcal{A}(\cdot,n)$, we conclude that there exists $\delta>0$ such that for any $y\in X$ with $\|x-y\|<\delta$, we have that
$$\sum_{n=0}^N \mathcal{N}(f^n(q),\|\mathcal{A}(q,n)y\|)>k.$$
Hence, $y\in X_{q,k}$ and we conclude that $X_{q,k}$ is open in $X$.

Let us now show that each $X_{q,k}$ is dense in $X$. Take $x\in X$ and $\delta>0$. We claim that $B(x,\delta)\cap X_{q,k}\neq\emptyset$. Assume the opposite and take an arbitrary $y\in X$ such that $\|y\|<\delta$. Then, $x+y\in B(x,\delta)$ and consequently,
$$\sum_{n=0}^\infty \mathcal{N}(f^n(q),\|\mathcal{A}(q,n)(x+y)\|)\leq k. \tag{26}$$

Furthermore, since $x \in B(x,\delta)$ we have that

$$\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)x\|) \le k. \tag{27}$$

Thus, using the third condition in the definition of $\mathcal{F}'(M)$, (26) and (27) it follows that

$$\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)y\|) < \infty.$$

Hence, $B(0,\delta) \subset X \setminus X_q$. This together with the third condition in the definition of $\mathcal{F}'(M)$ implies that $X_q = \emptyset$ which contradicts (25). □

The following result is similar in nature to Theorem 5 but it concerns different class of maps $\mathcal{N}$ and is valid under stronger assumption that $\lambda_\mu(\mathcal{A}) > 0$. Let $E \subset M$ be any Borel-measurable set and consider a family $\mathcal{F}''(E)$ that consists of all $\mathcal{N} \in \mathcal{F}(E)$ such that:

(1) $\mathcal{N} \colon M \times [0,\infty) \to [0,\infty)$ is a measurable map;
(2) $\mathcal{N}(q,\cdot)$ is lower semicontinuous for each $q \in E$;
(3) for each $q \in E$ and for infinite $I \subset \mathbb{N}$, we have that

$$\sum_{n\in I} \mathcal{N}(f^n(q), 1) = +\infty.$$

**Theorem 6.** *Assume that $\lambda_\mu(\mathcal{A}) > 0$ and take an arbitrary $\mathcal{N} \in \mathcal{F}''(M)$. Then, there exists a Borel-measurable set $E \subset M$ satisfying $\mu(E) = 1$ and such that $X_q$ given by (24) is a residual set for each $q \in E$.*

*Proof.* Using the same notation as in the proof of Theorem 5, we have that $X_{q,k}$ is open for each $q \in E$ and $k \in \mathbb{N}$. In addition, without any loss of assumption we may assume that

$$\lambda_\mu(\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)\| > 0, \quad \text{for every } q \in E. \tag{28}$$

It remains to prove that $X_{q,k}$ is also dense for $q \in E$ and $k \in \mathbb{N}$. Take now an arbitrary $x \in X$ and $\delta > 0$. We claim that there exists $y \in B(x,\delta)$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)y\| > 0. \tag{29}$$

If $y = x$ satisfies (29) there is nothing to prove. Otherwise, we have that

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)x\| \le 0. \tag{30}$$

On the other hand, it follows from (28) and [16, Proposition 14.] that there exists $z \in B(0,\delta)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)z\| > 0. \tag{31}$$

It follows readily from (30) and (31) that $y = x + z$ satisfies (29). Hence, we have found $y \in B(x,\delta)$ such that (29) holds. In particular, there exists an infinite subset $I$ of $\mathbb{N}$ such that

$$\|\mathcal{A}(q,n)y\| \ge 1, \quad \text{for each } n \in I.$$

This together with our assumption that $\mathcal{N} \in \mathcal{F}''(M)$ implies that

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)y\|) &\geq \sum_{n\in I} \mathcal{N}(f^n(q), \|\mathcal{A}(q,n)y\|) \\ &\geq \sum_{n\in I} \mathcal{N}(f^n(q), 1) \\ &= +\infty.\end{aligned}$$

We conclude that $y \in X_q$ and the proof is completed. □

**Remark 4.** *We note that similar results to our Theorems 5 and 6 have been obtained earlier by van Neerven [43] in the autonomous case, i.e. for semigroups of linear operators.*

2.3. **Nonuniform exponential instability.** Given $L \in B(X)$, we define the *mininorm* of $L$ as

$$\mathfrak{m}(L) = \inf\{\| Lv \| / \| v \| : v \neq 0\}.$$

It follows from Kingman's subadditive ergodic theorem [20] that there exists $\lambda_\mu^-(\mathcal{A}) \in [-\infty, \infty)$ such that

$$\lambda_\mu^-(\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log \mathfrak{m}(\mathcal{A}(q,n)) \quad \text{for } \mu\text{-a.e. } q \in M. \tag{32}$$

Such number is called the *smallest Lyapunov exponent* of $\mathcal{A}$ with respect to $\mu$. It is easy to see that, whenever $A(q)$ is an invertible operator for $\mu$-almost every $q \in M$,

$$\lambda_\mu^-(\mathcal{A}) = -\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)^{-1}\| \quad \text{for } \mu\text{-a.e. } q \in M.$$

Indeed, this follows from a simple observation that for any invertible operator $L \in B(X)$ we have that $\mathfrak{m}(L) = \|L^{-1}\|^{-1}$.

We now formulate conditions (in the spirit of Theorem 1) under which the smallest Lyapunov exponent of a given cocycle (and with respect to some ergodic invariant measure) is strictly positive.

**Theorem 7.** *For any $\mu \in \mathcal{E}(f)$, the following properties are equivalent:*

(1) $\lambda_\mu^-(\mathcal{A}) > 0$;
(2) *there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) = 1$, a Borel-measurable function $C\colon E \to (0,\infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that*

$$\sum_{n=0}^{\infty} \mathcal{N}\bigg(f^n(q), \frac{1}{\|\mathcal{A}(q,n)x\|}\bigg) \leq C(q)\mathcal{N}\bigg(q, \frac{1}{\|x\|}\bigg), \tag{33}$$

*for $q \in E$ and $x \in X \setminus \{0\}$;*
(3) *there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) > 0$, a Borel-measurable function $C\colon E \to (0,\infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that (33) holds for $q \in E$ and $x \in X \setminus \{0\}$.*

*Proof.* We start by proving that (1) implies (2). Assume that $\lambda_\mu^-(\mathcal{A}) > 0$ and let $\varepsilon > 0$ be such that $\lambda_\mu^-(\mathcal{A}) - \varepsilon > 0$. It follows from (32) that there exists a Borel-measurable set $E \subset M$ satisfying $\mu(E) = 1$ and such that

$$K(q) = \sup_n \left\{ \frac{e^{n(\lambda_\mu^-(\mathcal{A})-\varepsilon)}}{\mathfrak{m}(\mathcal{A}(q,n))} \right\} < +\infty$$

for every $q \in E$. Furthermore, $K$ is obviously a measurable map. Moreover,

$$\frac{1}{\mathfrak{m}(\mathcal{A}(q,n))} \leq K(q) e^{-n(\lambda_\mu^-(\mathcal{A})-\varepsilon)}$$

for every $q \in E$. Observing that

$$\frac{1}{\mathfrak{m}(\mathcal{A}(q,n))} = \sup_{\|x\| \neq 0} \frac{\|x\|}{\|\mathcal{A}(q,n)x\|},$$

we obtain

$$\frac{1}{\|\mathcal{A}(q,n)x\|} \leq \frac{K(q)e^{-n(\lambda_\mu^-(\mathcal{A})-\varepsilon)}}{\|x\|} \quad \text{for every } x \in X \setminus \{0\}.$$

In particular, for any $p > 0$ we have

$$\sum_{n=0}^{\infty} \frac{1}{\|\mathcal{A}(q,n)x\|^p} \leq \frac{1}{\|x\|^p} K(q)^p \sum_{n=0}^{\infty} e^{-n(\lambda_\mu^-(\mathcal{A})-\varepsilon)p}.$$

Consequently, setting

$$C(q) = K(q)^p \sum_{n=0}^{\infty} e^{-n(\lambda_\mu^-(\mathcal{A})-\varepsilon)p} = \frac{K(q)^p}{1 - e^{-(\lambda_\mu^-(\mathcal{A})-\varepsilon)p}},$$

we conclude that (33) holds with $\mathcal{N}(q,t) = t^p$.

The fact that (2) implies (3) is trivial so it remains to prove that (3) implies (1). Assume that there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) > 0$, a Borel-measurable function $C\colon E \to (0,\infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that (33) holds for $q \in E$ and $x \in X \setminus \{0\}$. As in the proof of Theorem 1, we may to assume that $E$ is compact and that $C$, $\mathcal{N}(\cdot, \frac{1}{2})$ and $\mathcal{N}(\cdot, 1)$ are continuous on $E$. For $q \in E$ and $x \in X \setminus \{0\}$, let

$$\|x\|_q := \sum_{n=0}^{\infty} \mathcal{N}\left(f^n(q), \frac{1}{\|\mathcal{A}(q,n)x\|}\right).$$

By (33),

$$\mathcal{N}\left(q, \frac{1}{\|x\|}\right) \leq \|x\|_q \leq C(q)\mathcal{N}\left(q, \frac{1}{\|x\|}\right), \tag{34}$$

for $q \in E$ and $x \in X \setminus \{0\}$. Since $E$ is compact and $C$ continuous on $E$, we have that there exists $K > 1$ such that

$$\sup_{q \in E} C(q) = \max_{q \in E} C(q) \leq K. \tag{35}$$

By repeating the arguments in the proof of Lemma 1, one can verify that

$$\|\mathcal{A}(q,m)x\|_{f^m(q)} \leq \gamma \|x\| \quad \text{for } q \in E \text{ and } x \in X \setminus \{0\},$$

where $\gamma := 1 - 1/K$. Let $E'$, $\tau$, $\bar{f}$ and $\bar{\mathcal{A}}$ be as in the proof of Theorem 1. Arguing as in the proof of Lemma 2, one can show that

$$\|\overline{\mathcal{A}}(q,n)x\|_{\overline{f}^n(q)} \leq \gamma^n \|x\|_q, \tag{36}$$

for $q \in E'$, $n \in \mathbb{N}$ and $x \in X \setminus \{0\}$. Observe that (34), (35) and (36) imply that

$$\mathcal{N}\left(\overline{f}^n(q), \frac{1}{\|\overline{\mathcal{A}}(q,n)x\|}\right) \le K\gamma^n \mathcal{N}(q,1),$$

for $q \in E'$, $n \in \mathbb{N}$ and $x \in X$ such that $\|x\| = 1$. As in the proof of Theorem 1 this implies that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{\|\mathcal{A}(q,\tau_{n_0}(q))x\|} = \frac{1}{\|\overline{\mathcal{A}}(q,n_0)x\|} \le \frac{1}{2} \quad \text{for } q \in E' \text{ and } \|x\| = 1,$$

and therefore

$$\frac{\|x\|}{\|\mathcal{A}(q,\tau_{kn_0}(q))x\|} \le \frac{1}{2^k} \quad \text{for } q \in E',\ k \in \mathbb{N} \text{ and } x \in X \setminus \{0\}.$$

In particular,

$$\mathfrak{m}(\mathcal{A}(q,\tau_{kn_0}(q))) \ge 2^k \text{ for every } k \in \mathbb{N} \text{ and } q \in E'.$$

Hence,

$$\begin{aligned}
\lambda_\mu^-(\mathcal{A}) &= \lim_{n\to\infty} \frac{1}{n} \log \mathfrak{m}(\mathcal{A}(q,n)) = \lim_{n\to\infty} \frac{1}{\tau_n(q)} \log \mathfrak{m}(\mathcal{A}(q,\tau_n(q))) \\
&= \lim_{k\to\infty} \frac{1}{\tau_{kn_0}(q)} \log \mathfrak{m}(\mathcal{A}(q,\tau_{kn_0}(q))) \ge \lim_{k\to\infty} \frac{1}{\tau_{kn_0}(q)} \log 2^k \\
&= \log 2 \cdot \lim_{k\to\infty} \frac{k}{\tau_{kn_0}(q)}.
\end{aligned}$$

Arguing as in the proof of Theorem 1 we have that

$$\lim_{k\to\infty} \frac{k}{\tau_{kn_0}(q)} > 0,$$

and thus the proof is completed. □

The proof of the following result can be established by repeating the arguments in the proof of Theorem 2 (see [15, Theorem 3]).

**Theorem 8.** *Assume that $\lambda_\mu^-(\mathcal{A}) > 0$ for some $\mu \in \mathcal{E}(f)$. Then, for each $\varepsilon > 0$ there exists a measurable function $T\colon M \to (0,\infty)$ such that:*

1. *for $\mu$-a.e. $q \in M$, $x \in X$ and $n \in \mathbb{N}_0$,*
$$\|\mathcal{A}(q,n)x\| \ge \frac{1}{T(q)} e^{(\lambda_\mu^-(\mathcal{A})-\varepsilon)n}\|x\|;$$
2. *for $\mu$-a.e. $q \in M$ and $n \in \mathbb{N}_0$,*
$$T(f^n(q)) \le T(q)e^{\varepsilon n}.$$

The following is a version of Theorem 3 for exponential instability. We omit the proof since it is analogous to the proof of Theorem 3.

**Theorem 9.** *Assume that $\mathcal{A}$ is an invertible cocycle over $f$ such that the map $A$ given by (4) is injective and continuous. Then, the following properties are equivalent:*

1. *there exist a full-measure set $E \subset M$, a Borel-measurable map $C\colon E \to (0,\infty)$ and $\mathcal{N} \in \mathcal{F}(E)$ such that (33) holds for each $q \in E$ and $x \in X$;*

(2) $\mathcal{A}$ *is uniformly exponentially unstable, i.e. there exist* $D, \lambda > 0$ *such that*

$$\|\mathcal{A}(q,n)x\| \geq \frac{1}{D} e^{\lambda n} \|x\| \quad \text{for every } q \in M,\ x \in X \text{ and } n \in \mathbb{N}_0.$$

Finally, we also state (without the proof) the following version of Theorem 5 in the current setting.

**Theorem 10.** *Assume that* $\lambda_\mu^-(\mathcal{A}) \leq 0$ *and take an arbitrary* $\mathcal{N} \in \mathcal{F}'(M)$ *as in Theorem 5. Then, there exists a Borel-measurable set* $E \subset M$ *satisfying* $\mu(E) = 1$ *and such that*

$$X_q := \left\{ x \in X : \sum_{n=0}^{\infty} \mathcal{N}\left( f^n(q), \frac{1}{\|\mathcal{A}(q,n)x\|} \right) = +\infty \right\}$$

*is a residual set, for each* $q \in E$.

Assume now that $X = \mathbb{R}^d$ and that $\mathcal{A}$ is a cocycle of (bounded) linear operators on $\mathbb{R}^d$. We will now formulate conditions under which all Lyapunov exponents of $\mathcal{A}$ (given by the Oseledets multiplicative ergodic theorem [4, 25]) belong to a given open interval.

**Theorem 11.** *Take* $\mu \in \mathcal{E}(f)$ *and assume that there exist a Borel-measurable set* $E \subset M$ *satisfying* $\mu(E) > 0$*, a Borel-measurable function* $C \colon E \to (0, \infty)$ *and* $\mathcal{N}_i \in \mathcal{F}(E)$*,* $i = 1, 2$ *such that*

$$\sum_{n=0}^{\infty} \mathcal{N}_1(f^n(q), e^{-an} \|\mathcal{A}(q,n)x\|) \leq C(q) \mathcal{N}_1(q, \|x\|) \tag{37}$$

*and*

$$\sum_{n=0}^{\infty} \mathcal{N}_2\left( f^n(q), \frac{1}{e^{-bn}\|\mathcal{A}(q,n)x\|} \right) \leq C(q) \mathcal{N}_2\left( q, \frac{1}{\|x\|} \right), \tag{38}$$

*for* $q \in E$ *and* $x \in X \setminus \{0\}$*, where* $b < a$*. Then, all Lyapunov exponents of* $\mathcal{A}$ *are contained in the interval* $(b, a)$*.*

*Proof.* Let $\mathcal{B}$ be a cocycle over $f$ with the generator $B$ given by $B(q) = e^{-a} A(q)$, $q \in M$. It follows from (37) and Theorem 1 that $\lambda_\mu(\mathcal{B}) < 0$. Since $\lambda_\mu(\mathcal{B}) = -a + \lambda_\mu(\mathcal{A})$, we conclude that $\lambda_\mu(\mathcal{A}) < a$.

Similarly, (38) and Theorem 7 imply that $\lambda_\mu^-(\mathcal{A}) > b$ and the proof is completed.

□

2.4. **Tempered exponential dichotomy.** By combining results from previous sections, we are in a position to formulate sufficient conditions for the existence of *tempered exponential dichotomies* which were recently studied in [2, 47].

**Theorem 12.** *Take* $\mu \in \mathcal{E}(f)$ *and assume that there exist a Borel-measurable set* $E \subset M$ *satisfying* $\mu(E) = 1$*, a Borel measurable map* $C \colon E \to (0, \infty)$*,* $\mathcal{N}_i \in \mathcal{F}(E)$*,* $i \in \{s, u\}$*, and a measurable splitting*

$$X = E^s(q) \oplus E^u(q) \quad \text{for } q \in E, \tag{39}$$

*where* $E^s(q)$ *and* $E^u(q)$ *are closed subspace of* $X$ *satisfying:*

- $A(q)E^s(q) \subset E^s(f(q))$ *and* $A(q)E^u(q) = E^u(f(q))$ *for* $q \in E$*;*

- $A(q)|_{E^u(q)} \colon E^u(q) \to E^u(f(q))$ *is invertible for* $q \in E$*;*
- (7) *holds for* $\mathcal{N}_s$, $q \in E$ *and* $x \in E^s(q)$*;*
- (33) *holds for* $\mathcal{N}_u$, $q \in E$ *and* $x \in E^u(q)$.

*Then, there exists* $\lambda > 0$ *and for each* $\varepsilon > 0$ *a measurable function* $T \colon M \to (0, \infty)$ *such that:*

- *for* $\mu$*-a.e.* $q \in M$, $x \in E^s(q)$ *and* $n \in \mathbb{N}$,
$$\|\mathcal{A}(q,n)x\| \le T(q)e^{(-\lambda+\varepsilon)n}\|x\|;$$
- *for* $\mu$*-a.e.* $q \in M$, $x \in E^u(q)$ *and* $n \in \mathbb{N}$,
$$\|\mathcal{A}(q,n)x\| \ge \frac{1}{T(q)}e^{(\lambda-\varepsilon)n}\|x\|;$$
- *for* $\mu$*-a.e.* $q \in M$,
$$\angle(E^s(q), E^u(q)) \le T(q);$$
- *for* $\mu$*-a.e.* $q \in M$ *and* $n \in \mathbb{N}$,
$$T(f^n(q)) \le T(q)e^{\varepsilon n}.$$

*Proof.* In a view of Theorems 1, 2, 7 and 8 only the third assertion of the theorem requires some elaboration. However, this requires the use of some standard arguments and thus we only give a sketch of the proof. We first note that we can assume without any loss of assumption that

$$\|\mathcal{A}(q,n)\| \le T(q)e^{(c+\varepsilon)n} \quad \text{for } \mu\text{-a.e. } q \in M,$$

where $c = \lambda_\mu(\mathcal{A})$. One can now easily introduce a family of the so-called Lyapunov norms [3, 4] that transform all the above inequalities concerning the growth of $\mathcal{A}$ into uniform estimates. More precisely, we have the following lemma.

**Lemma 3.** *There exists a family of norms* $\|\cdot\|_q$, $q \in M$ *on* $X$ *satisfying:*

- *there is* $C > 0$ *such that*
$$\|x\| \le \|x\|_q \le CT(q)\|x\| \quad \text{for } x \in X \text{ and } \mu\text{-a.e. } q \in M; \tag{40}$$
- *for* $\mu$*-a.e.* $q \in M$, $x \in E^s(q)$ *and* $n \in \mathbb{N}$,
$$\|\mathcal{A}(q,n)x\|_{f^n(q)} \le Ce^{(-\lambda+\varepsilon)n}\|x\|_q;$$
- *for* $\mu$*-a.e.* $q \in M$, $x \in E^u(q)$ *and* $n \in \mathbb{N}$,
$$\|\mathcal{A}(q,n)x\|_{f^n(q)} \ge \frac{1}{C}e^{(\lambda-\varepsilon)n}\|x\|_q;$$
- *for* $\mu$*-a.e.* $q \in M$, $x \in X$ *and* $n \in \mathbb{N}$,
$$\|\mathcal{A}(q,n)x\|_{f^n(q)} \le Ce^{(c+\varepsilon)n}\|x\|_q$$

Let $P(q) \colon X \to E^s(q)$ and $Q(q) \colon X \to E^u(q)$ be the projections associated with the splitting (39). By using Lemma 3 and repeating the arguments in the proof of [3, Lemma 4.4.], one can show that there exists $D > 0$ such that

$$\|P(q)x\|_q \le D\|x\|_q \quad \text{and} \quad \text{and} \quad \|Q(q)x\|_q \le D\|x\|_q,$$

for $\mu$-a.e. $q \in M$ and $x \in X$. The above estimates together wth (40) imply the third assertion of the theorem. □

## 3. Cocycles over semi-flows

In this section we present versions of some of our previous results for cocycles over semi-flows. In fact, all of them have such versions but, since many translations to this setting are straightforward, we chose to present only a few of them for the purpose of illustration.

We continue to assume that $M$ is a compact metric space and that $X$ is a separable Banach space. A family of maps $\Phi = (\varphi_t)_{t \ge 0}$, $\varphi_t \colon M \to M$ is said to be a *semi-flow* if:

(1) $\varphi_t$ is a continuous map for each $t \ge 0$;
(2) $\varphi_0 = \mathrm{Id}$;
(3) $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for $t, s \ge 0$;
(4) the map $(q,t) \mapsto \varphi^t(q)$ is continuous on $M \times [0,\infty)$.

Furthermore, we say that the map $\mathcal{A} \colon M \times [0,\infty) \to B(X)$ is a *cocycle* over $\Phi = (\varphi_t)_{t \ge 0}$ if:

(1) $\mathcal{A}(q,0) = \mathrm{Id}$ for $q \in M$;
(2) $\mathcal{A}(q,t+s) = \mathcal{A}(\varphi_s(q),t)\mathcal{A}(q,s)$ for $q \in M$ and $t,s \ge 0$;
(3) $(q,t) \mapsto \mathcal{A}(q,t)x$ is a continuous map on $M \times [0,\infty)$ for each $x \in X$.

It follows easily from the uniform boundness principle and the assumption that $M$ is compact that $\mathcal{A}$ is exponentially bounded, i.e. that there $K, \omega > 0$ such that

$$\|\mathcal{A}(q,t)\| \le K e^{\omega t} \quad \text{for } q \in M \text{ and } t \ge 0. \tag{41}$$

Let $\mathcal{E}(\Phi)$ denote the space of all ergodic, $\Phi$-invariant Borel probability measures. It follows from Kingman's subadditive ergodic theorem [20] that for each $\mu \in \mathcal{E}(\Phi)$, there exists $\lambda_\mu(\mathcal{A}) \in [-\infty, \infty)$ such that

$$\lambda_\mu(\mathcal{A}) = \lim_{t \to \infty} \frac{1}{t} \log\|\mathcal{A}(q,t)\|, \quad \text{for } \mu\text{-a.e. } q \in M. \tag{42}$$

As for cocycles over maps, the number $\lambda_\mu(\mathcal{A})$ is called the *largest Lyapunov exponent* of the cocycle $\mathcal{A}$ with respect to $\mu$.

We now establish the version of Theorem 1 for cocycles over flows. Let $\bar{\mathcal{F}}(E)$ denote all $\mathcal{N} \in \mathcal{F}(E)$ depending only on the second coordinate.

**Theorem 13.** *For any $\mu \in \mathcal{E}(\Phi)$, the following properties are equivalent:*

(1) $\lambda_\mu(\mathcal{A}) < 0$;
(2) *there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) = 1$, a Borel-measurable function $C \colon E \to (0,\infty)$ and $\mathcal{N} \in \bar{\mathcal{F}}(E)$ such that*

$$\int_0^\infty \mathcal{N}(\|\mathcal{A}(q,t)x\|)\, dt \le C(q)\mathcal{N}(\|x\|) \tag{43}$$

*for $q \in E$ and $x \in X$;*

(3) *there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) > 0$, a Borel-measurable function $C \colon E \to (0,\infty)$ and $\mathcal{N} \in \bar{\mathcal{F}}(E)$ such that (43) holds for $q \in E$ and $x \in X$.*

*Proof.* Let us prove that (1) implies (2). Assume therefore that $\lambda_\mu(\mathcal{A}) < 0$ and take an arbitrary $\varepsilon > 0$ such that $\lambda_\mu(\mathcal{A}) + \varepsilon < 0$. It follows from (42) that

$$\overline{C}(q) := \sup\{\|\mathcal{A}(q,t)\| e^{-t(\lambda_\mu(\mathcal{A})+\varepsilon)} : t \ge 0\} < \infty, \tag{44}$$

for $\mu$-a.e. $q \in M$. Obviously,

$$\|\mathcal{A}(q,t)\| \le \overline{C}(q)e^{t(\lambda_\mu(\mathcal{A})+\varepsilon)} \quad \text{for } \mu\text{-a.e. } q \in M \text{ and } t \ge 0. \tag{45}$$

Take $p > 0$ and let $\mathcal{N} \in \bar{\mathcal{F}}(E)$ be given by $\mathcal{N}(t) = t^p$. Using (45) we have that

$$\int_0^\infty \|\mathcal{A}(q,t)x\|^p \, dt \le \overline{C}(q)^p \|x\|^p \int_0^\infty e^{pt(\lambda_\mu(\mathcal{A})+\varepsilon)} \, dt = \frac{\overline{C}(q)^p}{-p(\lambda_\mu(\mathcal{A})+\varepsilon)}\|x\|^p$$

for $\mu$-a.e. $q \in M$ and $x \in X$. Hence, (43) holds with

$$C(q) := \frac{\overline{C}(q)}{(-p(\lambda_\mu(\mathcal{A})+\varepsilon))^{1/p}}, \quad q \in M.$$

It remains to prove that (3) implies (1) since (2) trivially implies (3). Hence, suppose that there exist a Borel-measurable set $E \subset M$ satisfying $\mu(E) > 0$, a Borel-measurable function $C \colon E \to (0,\infty)$ and $\mathcal{N} \in \bar{\mathcal{F}}(E)$ such that (43) holds for $q \in E$ and $x \in X$. If follows from (41) that

$$\|\mathcal{A}(q,n+1)x\| \le \|\mathcal{A}(\varphi_t(q), n+1-t)\| \cdot \|\mathcal{A}(q,t)x\| \le Ke^\omega \|\mathcal{A}(q,t)x\|,$$

for $q \in M$, $n \in \mathbb{N}_0$, $t \in [n, n+1]$ and $x \in X$. Thus,

$$\mathcal{N}(\|\mathcal{A}(q,n+1)x\|) \le \int_n^{n+1} \mathcal{N}(\|\mathcal{A}(q,t)Tx\|) \, dt$$

and consequently

$$\sum_{n=1}^\infty \mathcal{N}(\|\mathcal{A}(q,n)x\|) \le \int_0^\infty \mathcal{N}(\|\mathcal{A}(q,t)Tx\|) \, dt,$$

for $q \in E$ and every $x \in X$, where $T = \max\{Ke^\omega, 1\}$. It follows from (43) that

$$\sum_{n=0}^\infty \mathcal{N}(\|\mathcal{A}(q,n)x\|) \le (C(q)+1)\mathcal{N}(T), \tag{46}$$

for $q \in E$ and every $x \in X$ such that $\|x\| = 1$. Note that the restriction of $\mathcal{A}$ to $M \times \mathbb{N}_0$ is a cocycle over $\varphi_1$ and that $\mu \in \mathcal{E}(\varphi_1)$. Hence, it follows from Theorem 1, Remark 2 and (46) that

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)\| < 0 \quad \text{for } \mu\text{-a.e. } q \in M,$$

which implies that

$$\lambda_\mu(\mathcal{A}) = \lim_{t\to\infty} \frac{1}{t} \log\|\mathcal{A}(q,t)\| = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(q,n)\| < 0.$$

□

The following is a continuous-time version of Theorem 3.

**Theorem 14.** *Assume that $\mathcal{A}$ is a continuous cocycle over $\Phi$. Furthermore, suppose that $M$ is a compact topological space. Then, the following properties are equivalent:*

(1) *there exist a full-measure set $E \subset M$, a Borel-measurable map $C \colon E \to (0,\infty)$ and $\mathcal{N} \in \bar{\mathcal{F}}(E)$ such that (43) holds for each $q \in E$ and $x \in X$;*

(2) $\mathcal{A}$ *is uniformly exponentially stable, i.e. there exist* $D, \lambda > 0$ *such that*

$$\|\mathcal{A}(q,t)\| \le De^{-\lambda t} \quad \text{for every } q \in M \text{ and } t \ge 0. \tag{47}$$

*Proof.* We shall show that (1) implies (2) since the converse is easy. It follows from Theorem 13 that

$$\lambda_\mu(\mathcal{A}) < 0 \quad \text{for every } \mu \in \mathcal{E}(\Phi). \tag{48}$$

For each $t \ge 0$, we define $F_t \colon M \to \mathbb{R} \cup \{-\infty\}$ by

$$F_t(q) = \log\|\mathcal{A}(q,t)\|, \quad q \in M.$$

Note that $F_t$ is upper semi-continuous and that

$$F_{t+s}(q) \le F_t(\varphi_s(q)) + F_s(q), \quad \text{for } q \in M \text{ and } t, s \ge 0.$$

It follows from (48) and [23, Theorem A.3.] that

$$\lim_{t\to\infty} \frac{1}{t} \log \max_{q\in M}\|\mathcal{A}(q,t)\| < 0,$$

which immediately implies (47). □

Similarly to (32), we define the *smallest Lyapunov exponent* of $\mathcal{A}$ with respect to $\mu$ as $\lambda_\mu^-(\mathcal{A}) \in [-\infty, \infty)$ such that

$$\lambda_\mu^-(\mathcal{A}) = \lim_{t\to\infty} \frac{1}{t} \log \mathfrak{m}(\mathcal{A}(q,t)) \quad \text{for } \mu\text{-a.e. } q \in M.$$

The existence of such number is once again ensured by Kingman's subadditive ergodic theorem [20]. Thus, proceeding as in the proof of Theorem 13 we can get a version of Theorem 7 for cocycles over semi-flows.

**Theorem 15.** *Assume that there exist* $D > 0$ *and* $\omega > 0$ *such that*

$$\mathfrak{m}(\mathcal{A}(q,t)) \ge \frac{1}{D} e^{-\omega t}, \quad \text{for } q \in M \text{ and } t \in [0,1].$$

*For any* $\mu \in \mathcal{E}(\Phi)$*, the following properties are equivalent:*

(1) $\lambda_\mu^-(\mathcal{A}) > 0$;

(2) *there exist a Borel-measurable set* $E \subset M$ *satisfying* $\mu(E) = 1$*, a Borel-measurable function* $C \colon E \to (0,\infty)$ *and* $\mathcal{N} \in \bar{\mathcal{F}}(E)$ *such that*

$$\int_0^\infty \mathcal{N}\left(\frac{1}{\|\mathcal{A}(q,t)x\|}\right) dt \le C(q)\mathcal{N}\left(\frac{1}{\|x\|}\right) \tag{49}$$

*for* $q \in E$ *and* $x \in X$;

(3) *there exist a Borel-measurable set* $E \subset M$ *satisfying* $\mu(E) > 0$*, a Borel-measurable function* $C \colon E \to (0,\infty)$ *and* $\mathcal{N} \in \bar{\mathcal{F}}(E)$ *such that* (49) *holds for* $q \in E$ *and* $x \in X$.

Consequently, we can also get a complete characterization of tempered exponential dichotomies for continuous-time cocycles.

**Theorem 16.** *Take* $\mu \in \mathcal{E}(\Phi)$ *and assume that there exist a Borel-measurable set* $E \subset M$ *satisfying* $\mu(E) = 1$*, a Borel measurable map* $C \colon E \to (0,\infty)$*,* $\mathcal{N}_i \in \bar{\mathcal{F}}(E)$*,* $i \in \{s,u\}$*, and a measurable splitting*

$$X = E^s(q) \oplus E^u(q) \quad \text{for } q \in E,$$

*where* $E^s(q)$ *and* $E^u(q)$ *are closed subspace of* $X$ *satisfying:*

- $\mathcal{A}(q,t)E^s(q) \subset E^s(\varphi_t(q))$ *and* $\mathcal{A}(q,t)E^u(q) = E^u(\varphi_t(q))$ *for* $q \in E$*;*
- $\mathcal{A}(q,t)|_{E^u(q)} \colon E^u(q) \to E^u(\varphi_t(q))$ *is invertible for* $q \in E$ *and there exist* $D, \omega > 0$ *such that*

$$\mathfrak{m}(\mathcal{A}(q,t)|_{E^u(q)}) \geq \frac{1}{D}e^{-\omega t}, \quad \text{for } q \in E \text{ and } t \in [0,1].;$$

- (43) *holds for* $\mathcal{N}_s$*,* $q \in E$ *and* $x \in E^s(q)$*;*
- (49) *holds for* $\mathcal{N}_u$*,* $q \in E$ *and* $x \in E^u(q)$*.*

*Then, there exist* $\lambda > 0$ *and for each* $\varepsilon > 0$ *a measurable function* $T \colon M \to (0, \infty)$ *such that:*

- *for* $\mu$*-a.e.* $q \in M$*,* $x \in E^s(q)$ *and* $t \geq 0$*,*

$$\|\mathcal{A}(q,t)x\| \leq T(q)e^{(-\lambda+\varepsilon)t}\|x\|;$$

- *for* $\mu$*-a.e.* $q \in M$*,* $x \in E^u(q)$ *and* $t \geq 0$*,*

$$\|\mathcal{A}(q,t)x\| \geq \frac{1}{T(q)}e^{(\lambda-\varepsilon)t}\|x\|;$$

- *for* $\mu$*-a.e.* $q \in M$*,*

$$\angle(E^s(q), E^u(q)) \leq T(q);$$

- *for* $\mu$*-a.e.* $q \in M$ *and* $t \geq 0$*,*

$$T(\varphi_t(q)) \leq T(q)e^{\varepsilon t}.$$

One can similarly establish continuous time versions of all other results we have established for cocycles over maps.

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