A (μ, ν)-DICHOTOMY SPECTRUM

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ABSTRACT. In this work, we introduce three notions of dichotomy spectrum based on general growth rates and describe their structure. Our results are applicable to nonautonomous linear systems acting on general Banach spaces having negative μ-index of compactness, a condition which is satisfied, for instance, by any sequence of compact operators. Moreover, for any possible form of the spectra, we present an explicit example exhibiting such spectrum. Furthermore, as an application, we obtain normal forms of certain nonautonomous systems. We emphasize that the classical Sacker-Sell spectrum can be obtained as a very particular case of our setting.

1. INTRODUCTION

The notion of an *exponential dichotomy*, introduced by Perron [21], plays an important role in the study of nonautonomous dynamical systems. Roughly speaking, a system is said to admit an exponential dichotomy if, at each moment of time, the phase space splits into two complementary directions such that along one of these directions we have exponential expansion with time, while in the other one we have exponential contraction. Associated with this notion we have that of the *Sacker-Sell spectrum* or *dichotomy spectrum*, introduced by Sacker and Sell [25] in the study of linear skew product flows with compact base and latter extended to several different settings [2, 5, 6, 15, 26]. In general terms, this spectrum consists of all real numbers for which an appropriate perturbation, determined by these numbers, of the original system does not admit an exponential dichotomy. This notion has proved to be useful in several contexts, like in the obtention of normal forms for nonautonomous difference and differential equations [8, 27, 28, 29, 32], and is by now reasonably well understood.

In the present work we aim to extend the study of dichotomy spectrum by considering a similar notion but now associated to a more general concept of dichotomy, namely, that of (μ, ν)-*dichotomy*. Similarly to what happens in the case of an exponential dichotomy, this notion also requires that the phase space splits (at each moment of time) into two complementary directions along which the dynamics contracts/expands, but here the rates of contraction/expansion are given by a general function μ and the nonuniformity of these contractions/expansions is measured using a function ν (for more on this concept, check out Section 2.1). Then, based on this notion of (μ, ν)-dichotomy, we will introduce three notions of dichotomy spectrum and describe their structure. Our results are applicable to

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nonautonomous linear systems acting on general Banach spaces having negative μ-*index of compactness*, a condition which is satisfied, for instance, by any sequence of compact operators. Moreover, for any possible form of the spectra, we present an explicit example exhibiting such spectrum. Furthermore, as an application, we obtain normal forms of certain nonautonomous systems. The importance of our results stems from our general framework. More precisely, we are able to treat in a unified manner various settings in which no similar result has been previously obtained and to recover and refine several known results. In particular, we observe that the classical Sacker-Sell spectrum can be obtained as a very particular case of our setting.

We would like to mention that our work was inspired by the works of Barreira, Dragičević and Valls [5, 6]. In these works, the authors introduce and characterize a strong nonuniform spectrum associated with arbitrary growth rates for *finite-dimensional* systems and a nonuniform dichotomy spectrum associated with a nonuniform exponential dichotomy with an arbitrarily small nonuniform part for possible *infinite-dimensional* systems acting on Banach spaces and with index of compactness smaller than zero, respectively. In the present paper, we combine the main features of both works and deal with very general types of dichotomy (even more general than the one considered in [5]) and also work in the infinite-dimensional setting of Banach spaces. As for the application to normal forms, we mention that our work was inspired by [8] in which the authors have obtained normal forms by making use of the nonuniform dichotomy spectrum introduced in [6]. Here, we use our nonuniform μ-dichotomy spectrum to obtain a similar application. Finally, we refer to [1, 2, 15, 17, 23, 24, 25, 31, 32] and references therein for more interesting results involving several types of dichotomy spectra.

2. Preliminaries

Let $X = (X, \|\cdot\|)$ be an arbitrary Banach space. By $\mathcal{B}(X)$ we will denote the space of all bounded linear operators on X. The operator norm on $\mathcal{B}(X)$ will be also denoted by $\|\cdot\|$. Given a sequence $(A_n)_{n\in\mathbb{Z}}$ of bounded linear operators in $\mathcal{B}(X)$, let us consider the associated linear difference equation

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{2.1}$$

For $m, n \in \mathbb{Z}$, the evolution operator associated to (2.1) is given by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{for } m > n; \\ \mathrm{Id} & \text{for } m = n, \end{cases} \tag{2.2}$$

where Id denotes the identity operator on X.

2.1. Growth rates and (μ,ν)-dichotomy.

Let $\mu = (\mu_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence of positive numbers such that

$$\lim_{n\to-\infty} \mu_n = 0 \quad \text{and} \quad \lim_{n\to+\infty} \mu_n = +\infty. \tag{2.3}$$

We call such a sequence μ a *growth rate*. Furthermore, let $\nu = (\nu_n)_{n\in\mathbb{Z}}$ be an arbitrary sequence with $\nu_n \geq 1$ for every $n \in \mathbb{Z}$.

Definition 2.1. We say that (2.1) admits a (μ,ν)-*dichotomy* if the following conditions are satisfied:

(1) there exists a family of projections P_n, $n \in \mathbb{Z}$, such that

$$A_n P_n = P_{n+1} A_n; \tag{2.4}$$

(2) the restriction

$$A_n|_{\operatorname{Ker} P_n}\colon \operatorname{Ker} P_n \to \operatorname{Ker} P_{n+1} \tag{2.5}$$

is an invertible operator for each $n \in \mathbb{Z}$;

(3) there exist $D, \lambda > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \quad \text{for } m \ge n \tag{2.6}$$

and

$$\|\mathcal{A}(m,n)(\operatorname{Id} - P_n)\| \le D\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda} \quad \text{for } m \le n \tag{2.7}$$

where

$$\mathcal{A}(m,n) := \left(\mathcal{A}(n,m)|_{\operatorname{Ker} P_m}\right)^{-1}\colon \operatorname{Ker} P_n \to \operatorname{Ker} P_m, \tag{2.8}$$

for $m \le n$.

Remark 2.2. We would like to emphasize the great generality of the notion of (μ,ν)-dichotomy. For instance, suppose initially that $\nu_n = C$ for every $n \in \mathbb{Z}$ and some $C \ge 1$. Then, by taking $\mu_n = e^n$, $n \in \mathbb{Z}$, we recover the well-known notion of *exponential dichotomy*; by taking $\mu_n = 1 + n$, for $n \ge 0$ and $\mu_n = 1/(1-n)$ for $n < 0$, we get the notion of *polynomial dichotomy*; by taking $\mu_n = \ln(e+n)$ for $n \ge 0$ and $\mu_n = 1/\ln(e-n)$ for $n < 0$, we get the notion of *logarithmic dichotomy*. Moreover, in all these cases, if we take a general sequence ν instead of the constant one, for instance, $\nu_n = \mu_n^{\operatorname{sgn}(\mu_n - 1)\varepsilon}$ for some small $\varepsilon > 0$ and $n \in \mathbb{Z}$ where $\operatorname{sgn}(\mu_n - 1)$ denotes the sign of $\mu_n - 1$, we get nonuniform versions of those dichotomies.

Remark 2.3. We observe that versions of (μ,ν)-dichotomy for discrete and continous time dynamics have already appeared in the literature and have been investigated by many authors. For instance, among the topics that have already been explored for systems exhibiting this type of behaviour are invariant manifolds [9, 10, 11, 20], the shadowing property [3], admissibility [4, 30], reducibility [12, 22, 31] and roughness [19, 14, 16]. Moreover, spectral properties associated with variations of this notion were also studied, for instance, in [5, 12, 31]. We will compare these results with ours in a more systematic way throughout the text.

2.2. **(μ,ν)-dichotomy spectrum.** We define the (μ,ν)-*dichotomy spectrum* of (2.1) as the set of all numbers $\gamma \in \mathbb{R}$ for which the system

$$x_{n+1} = \left(\frac{\mu_n}{\mu_{n+1}}\right)^{\gamma} A_n x_n, \quad n \in \mathbb{Z}, \tag{2.9}$$

does not admit a (μ,ν)-dichotomy and denote this set by $\Sigma_{\mu,\nu}$. The set $\rho_{\mu,\nu} := \mathbb{R} \setminus \Sigma_{\mu,\nu}$ is called the (μ,ν)-*resolvent* set of (2.1). We will denote the evolution operator associated with (2.9) by $\mathcal{A}_\gamma(m,n)$. In particular,

$$\mathcal{A}_\gamma(m,n) = \left(\frac{\mu_n}{\mu_m}\right)^{\gamma} \mathcal{A}(m,n).$$

Remark 2.4. We observe that the classical Sacker-Sell spectrum can be recovered in the particular case when $\mu_n = e^n$ and $\nu_n = C$ for some $C \ge 1$ and all $n \in \mathbb{Z}$.

2.3. **μ-index of compactness.** Let B_X denote the closed unit ball in X centered at 0. For an arbitrary $A \in \mathcal{B}(X)$, let $\|A\|_{\mathrm{ic}}$ be the infimum over all $r > 0$ with the property that $A(B_X)$ can be covered by finitely many open balls of radius r. It is easy to show that

$$\|A\|_{\mathrm{ic}} \le \|A\| \quad \text{and} \quad \|cA\|_{\mathrm{ic}} = |c|\|A\|_{\mathrm{ic}},$$

for every $A \in \mathcal{B}(X)$ and $c \in \mathbb{R}$. Moreover,

$$\|A_1 A_2\|_{\mathrm{ic}} \leq \|A_1\|_{\mathrm{ic}} \cdot \|A_2\|_{\mathrm{ic}}, \quad \text{for every } A_1, A_2 \in \mathcal{B}(X).$$

Using $\|\cdot\|_{\mathrm{ic}}$ we define

$$\kappa_{\mathrm{ic}} := \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \|\mathcal{A}(n,0)\|_{\mathrm{ic}}.$$

We call this number the *μ-index of compactness* of (2.1). An interesting property of (2.1) involving the μ-index of compactness is the following.

Proposition 2.5. *Suppose that (2.1) admits a (μ,ν)-dichotomy with projections $(P_n)_{n\in\mathbb{Z}}$ and let $\lambda > 0$ be such that (2.6) and (2.7) are satisfied. Moreover, suppose there exists $\varepsilon \in (0, \lambda]$ such that*

$$\lim_{n\to+\infty} \mu_n^{-\varepsilon} \nu_n = 0. \tag{2.10}$$

Then, if $\kappa_{ic} < 0$ we have that $\dim \operatorname{Ker} P_n < +\infty$ for every $n \in \mathbb{Z}$.

Proof. We start observing that by the invertibility required in (2.5), $\dim \operatorname{Ker} P_n = \dim \operatorname{Ker} P_m$ for every $m, n \in \mathbb{Z}$. Thus, all we have to do is to prove that $\dim \operatorname{Ker} P_0 < +\infty$. Suppose that this is not the case and consider the unit ball centered at 0 in $\operatorname{Ker} P_0$ given by

$$B_{\operatorname{Ker} P_0} = \{v \in \operatorname{Ker} P_0 : \|v\| \leq 1\} = B_X \cap \operatorname{Ker} P_0.$$

By Riez's Lemma, there exists a sequence of elements $(v_k)_{k\in\mathbb{N}}$ in $B_{\operatorname{Ker} P_0}$ such that $\|v_k - v_j\| \geq 1/2$ for every $k \neq j$. Then, it follows by (2.7) that for $n \geq 0$,

$$\|v_k - v_j\| \leq D\nu_n \left(\frac{\mu_n}{\mu_0}\right)^{-\lambda} \|\mathcal{A}(n,0)(v_k - v_j)\|.$$

Therefore,

$$\|\mathcal{A}(n,0)(v_k - v_j)\| \geq \frac{1}{D\nu_n} \|v_k - v_j\| \left(\frac{\mu_n}{\mu_0}\right)^{\lambda} \geq \frac{1}{2D\nu_n} \left(\frac{\mu_n}{\mu_0}\right)^{\lambda}.$$

In particular, $\mathcal{A}(n,0)(B_X)$ cannot be covered by finitely many balls of radius $\frac{1}{4D\nu_n}\left(\frac{\mu_n}{\mu_0}\right)^{\lambda}$ which implies that $\|\mathcal{A}(n,0)\|_{\mathrm{ic}} \geq \frac{1}{4D\nu_n}\left(\frac{\mu_n}{\mu_0}\right)^{\lambda}$. Therefore,

$$\begin{aligned}
\kappa_{\mathrm{ic}} &= \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \|\mathcal{A}(n,0)\|_{\mathrm{ic}} \\
&\geq \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \left(\frac{1}{4D\nu_n}\left(\frac{\mu_n}{\mu_0}\right)^{\lambda}\right) \\
&= \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \left(\frac{\mu_n^{\lambda}}{\nu_n}\right).
\end{aligned}$$

Finally, given $\varepsilon \in (0, \lambda]$ such that (2.10) holds, there exists $C > 0$ satisfying $\nu_n \leq C\mu_n^{\varepsilon}$ for every $n \in \mathbb{N}$. Plugging this information into the previous inequality we get that

$$\kappa_{\mathrm{ic}} \geq \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \left(\frac{\mu_n^{\lambda}}{C\mu_n^{\varepsilon}}\right) \geq \lambda - \varepsilon \geq 0$$

contradicting our assumption that $\kappa_{\mathrm{ic}} < 0$. Thus, $\dim \operatorname{Ker} P_0 < +\infty$ and the proof is complete. ☐

Observe that whenever $\dim X < +\infty$ or, more generally, if A is a compact operator in $\mathcal{B}(X)$, then $\|A\|_{ic} = 0$. In particular, if $(A_n)_{n\in\mathbb{Z}}$ is a sequence such that A_n is a compact operator for *some* $n \in \mathbb{N}$, then $\mathcal{A}(m,0)$ is compact for every

$m \geq n$ and $\kappa_{\rm ic} = -\infty$ (taking $\log 0 = -\infty$). In this case, whenever (2.10) holds, the hypotheses of Proposition 2.5 are automatically satisfied.

In what follows, we are going to focus our attention in the description of $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = \Sigma_{\mu,\nu} \cap (\kappa_{\rm ic}, +\infty)$, that is, the restriction of the (μ,ν)-dichotomy spectrum to the set $(\kappa_{\rm ic}, +\infty)$. The reason for this is that, given $\gamma \in (\kappa_{\rm ic}, +\infty) \cap \rho_{\mu,\nu}$, we have that the μ-index of compactness associated with (2.9) is smaller than 0. Indeed,

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \|\mathcal{A}_\gamma(n,0)\|_{\rm ic} \\
&= \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \left\| \left(\frac{\mu_0}{\mu_n}\right)^\gamma \mathcal{A}(n,0) \right\|_{\rm ic} \\
&= \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \left(\frac{\mu_0}{\mu_n}\right)^\gamma + \limsup_{n\to+\infty} \frac{1}{\log \mu_n} \log \|\mathcal{A}(n,0)\|_{\rm ic} \\
&= -\gamma + \kappa_{\rm ic} < 0.
\end{aligned}$$

Thus, whenever (2.10) holds, by Proposition 2.5 we have that $\dim \operatorname{Ker} P_n < +\infty$ for every $n \in \mathbb{Z}$ where $(P_n)_{n\in\mathbb{Z}}$ is the family of projections associated with the (μ,ν)-dichotomy of (2.9).

3. The structure of the (μ,ν)-dichotomy spectrum

In this section, we are going to describe the possible structure of $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} := \Sigma_{\mu,\nu} \cap (\kappa_{\rm ic}, +\infty)$. For this purpose, let us consider the following possibilities:

(P1) $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = \emptyset$;

(P2) $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = (\kappa_{ic}, +\infty)$;

(P3) $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = I_1 \cup \bigcup_{j=2}^{k} [a_j, b_j]$ where $I_1 = [a_1, b_1]$ or $[a_1, +\infty)$ for some numbers
$$\kappa_{ic} < a_k \leq b_k < a_{k-1} \leq b_{k-1} < \ldots < a_1 \leq b_1 \tag{3.1}$$
and some $k \in \mathbb{N}$;

(P4) $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = I_1 \cup \bigcup_{j=2}^{k-1} [a_j, b_j] \cup (\kappa_{\rm ic}, b_k]$ where $I_1 = [a_1, b_1]$ or $[a_1, +\infty)$ and the numbers a_j and b_j are as in (3.1) for some $k \geq 2$. In the case when $k = 1$ we have $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = (\kappa_{\rm ic}, b_1]$;

(P5) $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = I_1 \cup \bigcup_{j=2}^{\infty} [a_j, b_j]$ where $I_1 = [a_1, b_1]$ or $[a_1, +\infty)$ for some numbers
$$\kappa_{ic} < \ldots < a_2 \leq b_2 < a_1 \leq b_1 \tag{3.2}$$
with $\lim_{j\to+\infty} a_j = \kappa_{\rm ic}$;

(P6) $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = I_1 \cup \bigcup_{j=2}^{\infty} [a_j, b_j] \cup (\kappa_{\rm ic}, b_\infty]$ where $I_1 = [a_1, b_1]$ or $[a_1, +\infty)$ and the numbers a_j and b_j are as in (3.2) with $b_\infty := \lim_{j\to+\infty} a_j > \kappa_{\rm ic}$.

Moreover, given $\gamma \in \mathbb{R}$ and $n \in \mathbb{Z}$, let us consider

$$S_\gamma(n) = \left\{ v \in X : \sup_{m\geq n} \left(\mu_m^{-\gamma} \|\mathcal{A}(m,n)v\|\right) < +\infty \right\}$$

and let $U_\gamma(n)$ be the space of all $v \in X$ for which there exists a sequence $(z_m)_{m\leq n}$ such that $z_n = v$, $z_m = A_{m-1} z_{m-1}$ for every $m \leq n$ and $\sup_{m\leq n} (\mu_m^{-\gamma} \|z_m\|) < +\infty$. It is easy to see that whenever $\gamma < \beta$,

$$S_\gamma(n) \subset S_\beta(n) \text{ and } U_\beta(n) \subset U_\gamma(n)$$

for every $n \in \mathbb{Z}$.

Finally, given subspaces $S, U \subset X$, let us consider

$$\angle(S,U) = \inf\{\|v - u\| : v \in S \text{ and } u \in U \text{ with } \|v\| = \|u\| = 1\}.$$

Observe that this quantity may be interpreted as the angle between the subspaces S and U.

3.1. **Main result.** The following is the main result of this section. In the statement, we use the adjective "admissible" for a number $j \in \mathbb{N}$ meaning "values of $j \in \mathbb{N}$ for which the expression makes sense according to the case we are dealing with".

Theorem 3.1. *Suppose $\kappa_{ic} < 0$ and that for every $\varepsilon > 0$,*

$$\lim_{n\to-\infty} \mu_n^{\varepsilon}\nu_n = 0 \quad \textit{and} \quad \lim_{n\to+\infty} \mu_n^{-\varepsilon}\nu_n = 0. \tag{3.3}$$

Then $\Sigma_{\mu,\nu}^{\kappa_{ic}}$ has one of the forms given in (P1)-(P6). Moreover, in the case when $\Sigma_{\mu,\nu} \cap \mathbb{R}^+$ is bounded, taking numbers $c_j \in (b_{j+1}, a_j)$ for each j and $c_0 > b_1$, the subspaces

$$E_j(n) = S_{c_{j-1}}(n) \cap U_{c_j}(n)$$

are finite-dimensional, independent of the choices of c_j and satisfy the following properties:

- *for every admissible $j \geq 1$ and $n \in \mathbb{Z}$,*

$$A_n|_{E_j(n)} \colon E_j(n) \to E_j(n+1) \tag{3.4}$$

  is invertible;
- *for every admissible $1 \leq i \leq j$ and $n \in \mathbb{Z}$,*

$$U_{c_j}(n) = U_{c_{i-1}}(n) \oplus \bigoplus_{l=i}^{j} E_l(n); \tag{3.5}$$

- *for each admissible $i \geq 1$ and $l \geq 0$ and $n \in \mathbb{Z}$,*

$$X = S_{c_{i+l}}(n) \oplus U_{c_{i-1}}(n) \oplus \bigoplus_{j=i}^{i+l} E_j(n); \tag{3.6}$$

- *for $v \in E_j(n) \setminus \{0\}$,*

$$a_j \leq \liminf_{m\to\pm\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \leq \limsup_{m\to\pm\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \leq b_j. \tag{3.7}$$

- *for every $i \neq j$,*

$$\lim_{n\to\pm\infty} \frac{1}{\log \mu_n} \log \angle(E_i(n), E_j(n)) = 0. \tag{3.8}$$

Remark 3.2. In Section 4 we will present examples showing that all the possibilities for $\Sigma_{\mu,\nu}^{\kappa_{\mathrm{ic}}}$ given in the previous theorem can actually appear. Moreover, we once more observe that whenever $(A_n)_{n\in\mathbb{Z}}$ is a sequence such that A_n is a compact operator for some $n \in \mathbb{N}$, $\kappa_{\mathrm{ic}} = -\infty$ and $\Sigma_{\mu,\nu}^{\kappa_{\mathrm{ic}}} = \Sigma_{\mu,\nu}$. In particular, in this context Theorem 3.1 gives us a complete characterization of the (μ,ν)-dichotomy spectrum of (2.1).

Before we go into the proof of Theorem 3.1, we will present a simple yet interesting observation which completely characterizes the *stable* and *unstable* spaces associated to a (μ,ν)-dichotomy.

Proposition 3.3. *Suppose (2.1) admits a (μ,ν)-dichotomy with projections $(P_n)_{n\in\mathbb{Z}}$ and constants $D, \lambda > 0$. Moreover, assume that*

$$\lim_{n\to-\infty} \mu_n^{\lambda}\nu_n = 0 \quad \textit{and} \quad \lim_{n\to+\infty} \mu_n^{-\lambda}\nu_n = 0. \tag{3.9}$$

Then,

$$\operatorname{Im} P_n = \left\{ v \in X : \sup_{m\geq n} \|\mathcal{A}(m,n)v\| < +\infty \right\}$$

while $\operatorname{Ker} P_n$ consists of all $v \in X$ for which there exists a sequence $(z_m)_{m\leq n}$ such that $z_n = v$, $z_m = A_{m-1}z_{m-1}$ for every $m \leq n$ and $\sup_{m\leq n} \|z_m\| < +\infty$.

Proof. Given $v \in \operatorname{Im} P_n$, condition (2.6) implies that

$$\sup_{m\geq n} \|\mathcal{A}(m,n)v\| < +\infty \tag{3.10}$$

showing that $\operatorname{Im} P_n \subset \{v \in X : \sup_{m\geq n} \|\mathcal{A}(m,n)v\| < +\infty\}$. To show the reverse inclusion, we start by observing that, if $v \in X$ is such that (3.10) holds, then by (2.6) we have that

$$\sup_{m\geq n} \|\mathcal{A}(m,n)(\operatorname{Id} - P_n)v\| < +\infty. \tag{3.11}$$

On the other hand, given $m \geq n$, condition (2.7) implies that

$$\|(\operatorname{Id} - P_n)v\| \leq D\nu_m \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \|\mathcal{A}(m,n)(\operatorname{Id} - P_n)v\|$$

and thus

$$\frac{1}{D\nu_m}\left(\frac{\mu_m}{\mu_n}\right)^{\lambda} \|(\operatorname{Id} - P_n)v\| \leq \|\mathcal{A}(m,n)(\operatorname{Id} - P_n)v\|.$$

Consequently, if $(\operatorname{Id} - P_n)v \neq 0$, using condition (3.9) we get that the right-hand side of the previous inequality goes to infinity as $m \to +\infty$ which contradicts (3.11). Thus, $(\operatorname{Id} - P_n)v = 0$ and $\operatorname{Im} P_n = \{v \in X : \sup_{m\geq n} \|\mathcal{A}(m,n)v\| < +\infty\}$.

To prove the second claim, given $v \in \operatorname{Ker} P_n$, by (2.5) we may consider the sequence $z_m = \mathcal{A}(m,n)v$ for $m \leq n$. Then, $(z_m)_{m\leq n}$ satisfies $z_{m+1} = A_m z_m$ for $m < n$ and $z_n = v$. Moreover, condition (2.7) gives us that $\sup_{m\leq n} \|z_m\| < +\infty$. Reciprocally, if $v \in X$ is such that there exists a sequence $(z_m)_{m\leq n}$ satisfying $z_n = v$, $z_m = A_{m-1} z_{m-1}$ for every $m \leq n$ and $\sup_{m\leq n} \|z_m\| < +\infty$ then by (2.4) and (2.6),

$$\|P_n v\| = \|\mathcal{A}(n,m)P_m z_m\| \leq D\nu_m \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda} \|z_m\|$$

for $m \leq n$. Thus, making $m \to -\infty$ and using (3.9) we get that $P_n v = 0$. This concludes the proof of the second claim as well as the proposition. □

Remark 3.4. Observe that condition (3.3) implies that condition (3.9) holds for any $\lambda > 0$.

Proof of Theorem 3.1. Given $\gamma \in \mathbb{R}$ and $n \in \mathbb{Z}$, let us consider the sets $S_\gamma(n)$ and $U_\gamma(n)$ defined at the beginning of this section. We have already observed that $S_\gamma(n) \subset S_\beta(n)$ and $U_\beta(n) \subset U_\gamma(n)$ for every $n \in \mathbb{Z}$ whenever $\gamma < \beta$. Moreover, by Proposition 3.3, for every $\gamma \in \rho_{\mu,\nu}$, the projections $(P_n)_{n\in\mathbb{Z}}$ associated to the (μ,ν)-dichotomy of (2.9) satisfy

$$\operatorname{Im} P_n = S_\gamma(n) \quad \text{and} \quad \operatorname{Ker} P_n = U_\gamma(n) \tag{3.12}$$

and

$$X = S_\gamma(n) \oplus U_\gamma(n) \tag{3.13}$$

for every $n \in \mathbb{Z}$. In particular, by condition (2.5), $\dim U_\gamma(n)$ does not depend on $n \in \mathbb{Z}$ and thus we will write $\dim U_\gamma := \dim U_\gamma(n)$ for any $n \in \mathbb{Z}$. Furthermore, as observed in the end of Section 2.3, $\dim U_\gamma < +\infty$ for every $\gamma \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$.

We now present a series of auxiliary results.

Lemma 3.5. *The set $\rho_{\mu,\nu}$ is open. Moreover, if $\gamma \in \rho_{\mu,\nu}$ and $J \subset \rho_{\mu,\nu}$ is an interval containing γ, then*

$$S_\gamma(n) = S_\beta(n) \quad \text{and} \quad U_\gamma(n) = U_\beta(n)$$

for every $\beta \in J$ and $n \in \mathbb{Z}$.

Proof of Lemma 3.5. Let $\gamma \in \rho_{\mu,\nu}$. Then, there exists a family of projections $(P_n)_{n\in\mathbb{Z}}$ and constants $D, \lambda > 0$ such that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\gamma} \mathcal{A}(m,n)P_n \right\| \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \quad \text{for } m \geq n$$

and

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\gamma} \mathcal{A}(m,n)(\mathrm{Id} - P_n) \right\| \leq D\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda} \quad \text{for } m \leq n.$$

Then, for each $\beta \in \mathbb{R}$ such that $|\gamma - \beta| < \lambda/4$, taking $\tilde{\lambda} = \min\{\lambda - \gamma + \beta, \lambda + \gamma - \beta\} > 0$ we have that

$$\begin{aligned}\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\beta} \mathcal{A}(m,n)P_n \right\| &\leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-(\lambda-\gamma+\beta)} \\ &\leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\tilde{\lambda}}\end{aligned}$$

for $m \geq n$ and

$$\begin{aligned}\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\beta} \mathcal{A}(m,n)(\mathrm{Id} - P_n) \right\| &\leq D\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-(\lambda+\gamma-\beta)} \\ &\leq D\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-\tilde{\lambda}}\end{aligned}$$

for $m \leq n$. Then, $\beta \in \rho_{\mu,\nu}$ and $\rho_{\mu,\nu}$ is open.

Finally, let $J \subset \rho_{\mu,\nu}$ be an interval containing γ and take $\beta \in J$. To fix notation, assume $\beta < \gamma$. By the previous argument, for each $\eta \in [\beta, \gamma]$ there exists an open interval I_η containing η such that $I_\eta \subset \rho_{\mu,\nu}$ and, for each $\xi \in I_\eta$, the projections given by the (μ,ν)-dichotomy associated to ξ and η are the same. Thus, it follows from (3.12) that $S_\eta(n) = S_\xi(n)$ and $U_\eta(n) = U_\xi(n)$ for every $n \in \mathbb{Z}$. Consequently, since the intervals I_η form an open cover of $[\beta, \gamma]$ and this is a compact interval, it follows that $S_\gamma(n) = S_\beta(n)$ and $U_\gamma(n) = U_\beta(n)$ for every $n \in \mathbb{Z}$. Similarly we can consider the case when $\beta > \gamma$. This concludes the proof of the lemma. □

Lemma 3.6. *Let $\gamma_1, \gamma_2 \in \rho_{\mu,\nu} \cap (\kappa_{ic}, +\infty)$ be such that $\gamma_1 < \gamma_2$. Then, $[\gamma_1, \gamma_2] \cap \Sigma_{\mu,\nu} \neq \emptyset$ if and only if $\dim U_{\gamma_2} < \dim U_{\gamma_1}$.*

Proof of Lemma 3.6. Take $\gamma_1, \gamma_2 \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$ with $\gamma_1 < \gamma_2$ and suppose initially that $[\gamma_1, \gamma_2] \cap \Sigma_{\mu,\nu} \neq \emptyset$. We have already observed that $U_{\gamma_2}(n) \subset U_{\gamma_1}(n)$ for every $n \in \mathbb{Z}$. Thus, either $\dim U_{\gamma_1} = \dim U_{\gamma_2}$ or $\dim U_{\gamma_2} < \dim U_{\gamma_1}$. Suppose that $\dim U_{\gamma_1} = \dim U_{\gamma_2}$. Then, $U_{\gamma_1}(n) = U_{\gamma_2}(n)$ for every $n \in \mathbb{Z}$ (recall that $\dim U_\gamma < +\infty$ for every $\gamma \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$). Consequently, by (3.12), $S_{\gamma_1}(n) = S_{\gamma_2}(n)$ for every $n \in \mathbb{Z}$ and there exists a family of projections $(P_n)_{n\in\mathbb{Z}}$ and constants $D_i, \lambda_i > 0$, $i = 1, 2$, such that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\gamma_i} \mathcal{A}(m,n)P_n \right\| \leq D_i\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda_i} \quad \text{for } m \geq n \tag{3.14}$$

and

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\gamma_i} \mathcal{A}(m,n)(\mathrm{Id} - P_n) \right\| \leq D_i\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda_i} \quad \text{for } m \leq n. \tag{3.15}$$

Then, for $\gamma \in [\gamma_1, \gamma_2]$, (3.14) for $i = 1$ gives us that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\gamma} \mathcal{A}(m,n)P_n \right\| \leq D_1\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda_1} \quad \text{for } m \geq n$$

while (3.15) for $i = 2$ gives us that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\gamma} \mathcal{A}(m,n)(\mathrm{Id} - P_n) \right\| \le D_2 \nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda_2} \quad \text{for } m \le n.$$

Therefore, taking $D = \max\{D_1, D_2\}$ and $\lambda = \min\{\lambda_1, \lambda_2\}$, it follows that (2.9) admits a (μ,ν)-dichotomy with constants D and λ. Hence, $\gamma \in \rho_{\mu,\nu}$. Thus, since $\gamma \in [\gamma_1, \gamma_2]$ was arbitrary, it follows that $[\gamma_1, \gamma_2] \subset \rho_{\mu,\nu}$, contradicting our assumption. Therefore, $\dim U_{\gamma_2} < \dim U_{\gamma_1}$ as claimed.

Suppose now that $\dim U_{\gamma_2} < \dim U_{\gamma_1}$ and take

$$\gamma := \inf\{\beta \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty) : \dim U_\beta = \dim U_{\gamma_2}\}.$$

Then, since $\dim U_{\gamma_1} > \dim U_{\gamma_2}$, it follows by Lemma 3.5 that $\gamma_1 < \gamma < \gamma_2$ which in particular implies that $\gamma > \kappa_{\mathrm{ic}}$. We will now show that $\gamma \in \Sigma_{\mu,\nu}$. Suppose that this is not the case. Then, we conclude by Lemma 3.5 that we can not have $\dim U_\gamma = \dim U_{\gamma_2}$ (otherwise this would contradict the definition of γ). Consequently, the only possibility is that $\dim U_{\gamma_2} < \dim U_\gamma$. In this case, again by Lemma 3.5, there exists $\varepsilon > 0$ such that $\gamma + \varepsilon < \gamma_2$, $[\gamma, \gamma+\varepsilon] \subset \rho_{\mu,\nu}$ and $\dim U_\gamma = \dim U_{\gamma+\varepsilon}$. In particular, $\dim U_{\gamma+\varepsilon} \neq \dim U_{\gamma_2}$ contradicting again the definition of γ. Therefore, $\gamma \in \Sigma_{\mu,\nu}$ and $[\gamma_1, \gamma_2] \cap \Sigma_{\mu,\nu} \neq \emptyset$. This concludes the proof of the lemma. □

Combining the previous results, we get the following fact.

Lemma 3.7. *If γ and β belong to the same connected component of $\rho_{\mu,\nu} \cap (\kappa_{ic}, +\infty)$, then $S_\gamma(n) = S_\beta(n)$ and $U_\gamma(n) = U_\beta(n)$ for every $n \in \mathbb{Z}$.*

We are now ready to conclude the proof of Theorem 3.1. We start by describing the structure of $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu}$. If $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu} = (\kappa_{\mathrm{ic}}, +\infty)$ then we are in the case (P2) and we are done. So, from now on suppose that $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu} \neq (\kappa_{\mathrm{ic}}, +\infty)$ and, consequently, $\rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty) \neq \emptyset$. By Lemma 3.5 we know that $\rho_{\mu,\nu}$ is an open subset of $(\kappa_{\mathrm{ic}}, +\infty)$ and thus it may be written as a finite or countable union of mutually disjoint open intervals. Consequently, $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu}$ is either empty or it can be written as a *finite* or *countable* union of mutually disjoint closed intervals of $(\kappa_{\mathrm{ic}}, +\infty)$. If $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu} = \emptyset$, then we are in the case (P1) and we are done. On the other hand, if $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu}$ can be written as a *finite* union of mutually disjoint closed intervals of $(\kappa_{\mathrm{ic}}, +\infty)$, then we are either in case (P3) or in case (P4) and again we are done. It remains to analyze the case where $\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu}$ can be written as a *countable* union of mutually disjoint closed intervals of $(\kappa_{\mathrm{ic}}, +\infty)$. For this purpose, we need the following observations.

Claim 1. For every $\gamma \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$, the set $\Sigma_{\mu,\nu} \cap [\gamma, +\infty)$ can be written as a finite union of mutually disjoint closed intervals. More precisely, if $d := \dim U_\gamma$ (recall that this is a finite number due to Proposition 2.5), then $\Sigma_{\mu,\nu} \cap [\gamma, +\infty)$ may be written as a finite union of at most $d+1$ mutually disjoint closed intervals.

Proof of Claim 1. For the sake of contradiction, suppose $\Sigma_{\mu,\nu} \cap [\gamma, +\infty)$ can be written as the union of at least $d+2$ mutually disjoint closed intervals. Then there exist real numbers $\gamma_1 < \gamma_2 < \ldots < \gamma_{d+1}$ in $\rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$ such that $[\gamma_i, \gamma_{i+1}] \cap \Sigma_{\mu,\nu} \neq \emptyset$ for every $i = 1, \ldots, d$. Consequently, by Lemma 3.6 we get that

$$d \ge \dim U_{\gamma_1} > \dim U_{\gamma_2} > \ldots > \dim U_{\gamma_{d+1}}$$

which is a contradiction and, therefore, the claim is proved. □

Claim 2. For every $\gamma_1 \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$, there exists $\gamma_2 \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, \gamma_1)$ such that $(\gamma_2, \gamma_1) \cap \Sigma_{\mu,\nu} \neq \emptyset$.

Proof of Claim 2. Suppose that the claim is false. Then, there exists $\gamma_1 \in \rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, +\infty)$ such that either $\rho_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, \gamma_1) = \emptyset$ and then

$$\Sigma^{\kappa_{\mathrm{ic}}}_{\mu,\nu} \cap (\kappa_{\mathrm{ic}}, \gamma_1) = (\kappa_{\mathrm{ic}}, \gamma_1)$$

or $(\kappa_{\text{ic}}, \gamma_1) \subset \rho_{\mu,\nu}$. In both cases, by applying Claim 1 to $\Sigma_{\mu,\nu} \cap [\gamma_1, +\infty)$, we get a contraction with the fact that $\Sigma_{\mu,\nu}^{\kappa_{\text{ic}}}$ can be written as a countable union of mutually disjoint closed intervals of $(\kappa_{\text{ic}}, +\infty)$. □

Using Claim 2 and proceeding inductively we obtain a decreasing sequence of numbers $(\gamma_n)_{n\in\mathbb{N}}$ satisfying

$$\gamma_n \in \rho_{\mu,\nu} \cap (\kappa_{\text{ic}}, +\infty) \;\text{ and }\; (\gamma_{n+1}, \gamma_n) \cap \Sigma_{\mu,\nu} \neq \emptyset.$$

Moreover, by Claim 1, for each $n \in \mathbb{N}$ there exists mutually disjoint closed intervals $I_1, I_2, \ldots, I_{k_n}$ such that

$$\Sigma_{\mu,\nu} \cap [\gamma_n, +\infty) = I_{k_n} \cup I_{k_n-1} \cup \ldots \cup I_1$$

where $(k_n)_{n\in\mathbb{N}}$ is an increasing sequence. Then, either $\lim_{n\to+\infty} \gamma_n = \kappa_{\text{ic}}$ and we are in case (P5) or $\lim_{n\to+\infty} \gamma_n =: b_\infty > \kappa_{\text{ic}}$. In the latter case, Claim 1 implies that $(\kappa_{\text{ic}}, b_\infty] \subset \Sigma_{\mu,\nu}$ and thus we are in case (P6). This concludes the description of the possible structure of $\Sigma_{\mu,\nu}^{\kappa_{\text{ic}}}$.

Let us now prove the remaining claims in Theorem 3.1. Fix $n \in \mathbb{Z}$. We start by observing that Lemma 3.5 implies that $E_j(n)$ does not depend neither on c_j nor on c_{j-1}. Moreover, given $i < j$, we have that

$$E_j(n) \subset S_{c_{j-1}}(n) \subset S_{c_i}(n) \;\text{ and }\; E_i(n) \subset U_{c_i}(n). \tag{3.16}$$

Thus, by (3.13) it follows that $E_i(n) \cap E_j(n) = \{0\}$. Furthermore, since

$$(U + V) \cap W = (U \cap W) + V$$

whenever U, V and W are subspaces with $V \subset W$, using (3.13) we get that for every j,

$$\begin{aligned} U_{c_j}(n) &= \big(S_{c_{j-1}}(n) \oplus U_{c_{j-1}}(n)\big) \cap U_{c_j}(n) \\ &= \big(S_{c_{j-1}}(n) \cap U_{c_j}(n)\big) \oplus U_{c_{j-1}}(n) \\ &= E_j(n) \oplus U_{c_{j-1}}(n). \end{aligned} \tag{3.17}$$

Proceeding recursively, we conclude that for every $1 \le i \le j$,

$$U_{c_j}(n) = U_{c_{i-1}}(n) \oplus \bigoplus_{l=i}^{j} E_l(n)$$

proving (3.5). Combining this observation with (3.13) we conclude that (3.6) holds.

Now, given $\gamma \in \mathbb{R}$, it follows easily from the definition that

$$A_n S_\gamma(n) \subset S_\gamma(n+1) \;\text{ and }\; A_n U_\gamma(n) \subset U_\gamma(n+1)$$

Thus, the definition of $E_j(n)$ readily implies that $A_n E_j(n) \subset E_j(n+1)$. Moreover, given $\gamma \in \rho_{\mu,\nu}$, it follows by Proposition 3.3, Eq. (3.12) and (2.5) that $A_n|_{U_\gamma(n)} \colon U_\gamma(n) \to U_\gamma(n+1)$ is invertible and, in particular, since $\dim U_{c_j}(n) < \infty$ for every j, we have $\dim U_{c_j}(n) = \dim U_{c_j}(n+1)$ for every j and $n \in \mathbb{Z}$. Thus, using (3.17) we conclude that $\dim E_j(n) = \dim E_j(n+1)$. Finally, recalling that $E_j(n) \subset U_{c_{j+1}}(n)$ and $A_n|_{U_{c_{j+1}}(n)}$ is injective, we get that $A_n|_{E_j(n)}$ is also injective and thus $A_n|_{E_j(n)} \colon E_j(n) \to E_j(n+1)$ is invertible proving (3.4). Let us now prove (3.7).

Let $v \in E_j(n) \setminus \{0\}$ with $j \ge 1$. Since $c_{j-1} \in \rho_{\mu,\nu}$, we have that (2.9) admits a (μ, ν)-dichotomy with $\gamma = c_{j-1}$. In particular, there exists a family of projections $(P_n)_{n\in\mathbb{Z}}$ and constants $D, \lambda > 0$ such that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{c_{j-1}} \mathcal{A}(m,n) P_n \right\| \le D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \qquad \text{for } m \ge n.$$

Thus, since $v \in S_{c_{j-1}}(n)$, by (3.12) we get that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{c_{j-1}} \mathcal{A}(m,n)v \right\| \le D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \|v\| \quad \text{for } m \ge n$$

which implies that

$$\limsup_{m\to+\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \le \limsup_{m\to+\infty} \frac{\log \mu_m^{-\lambda+c_{j-1}}}{\log \mu_m} = c_{j-1} - \lambda < c_{j-1}.$$

Making $c_{j-1} \searrow b_j$ we get that

$$\limsup_{m\to+\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \le b_j.$$

Similarly, since $c_j \in \rho_{\mu,\nu}$, we have that (2.9) admits a (μ,ν)-dichotomy with $\gamma = c_j$. In particular, there exists a family of projections $(P_n)_{n\in\mathbb{Z}}$ and constants $D, \lambda > 0$ such that

$$\left\| \left(\frac{\mu_m}{\mu_n}\right)^{c_j} \mathcal{A}(n,m)(\operatorname{Id} - P_m) \right\| \le D\nu_m \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \quad \text{for } m \ge n.$$

Thus, since $v \in U_{c_j}(n)$, by (3.12) and (2.5) we get that

$$\|v\| \le D\nu_m \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \left\| \left(\frac{\mu_n}{\mu_m}\right)^{c_j} \mathcal{A}(m,n)v \right\| \quad \text{for } m \ge n.$$

In particular,

$$\frac{1}{D\nu_m} \left(\frac{\mu_m}{\mu_n}\right)^{\lambda+c_j} \|v\| \le \|\mathcal{A}(m,n)v\| \quad \text{for } m \ge n.$$

Consequently, using (3.3) we get

$$\begin{aligned}\liminf_{m\to+\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| &\ge \liminf_{n\to+\infty} \frac{1}{\log \mu_m} \left(\log \mu_m^{\lambda+c_j} - \log D\nu_m\right) \\ &= c_j + \lambda \ge c_j.\end{aligned}$$

Thus, making $c_j \nearrow a_j$ it follows that

$$\liminf_{m\to+\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \ge a_j.$$

Similarly we can prove that

$$a_j \le \liminf_{m\to-\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \le \limsup_{m\to-\infty} \frac{1}{\log \mu_m} \log \|\mathcal{A}(m,n)v\| \le b_j.$$

Combining these observations we get that (3.7) is satisfied. It remains to observe that (3.8) also holds.

We may assume without loss of generality that $i < j$. Then, it is easy to see that

$$\angle(E_i(n), E_j(n)) \le 2.$$

Moreover, by (3.16) we have that $E_j(n) \subset S_{c_i}(n)$ and $E_i(n) \subset U_{c_i}(n)$. Now, letting $(P_n)_{n\in\mathbb{Z}}$ be the family of projections associated to (2.9) with $\gamma = c_i$, it follows by [7, Proposition 2.4] and (3.12) that

$$\frac{1}{\|P_n\|} \le \angle(S_{c_i}(n), U_{c_i}(n))$$

for every $n \in \mathbb{Z}$. Combining these observations with (2.6) for $m = n$ we get that

$$\frac{1}{D\nu_n} \le \angle(E_i(n), E_j(n)) \le 2$$

for every $n \in \mathbb{Z}$. Then, using (2.3) and (3.3) we get that

$$\lim_{n\to\pm\infty} \frac{1}{\log \mu_n} \log \angle(E_i(n), E_j(n)) = 0$$

as claimed. This concludes the proof the theorem. $\square$

Corollary 3.8. *Suppose* $\dim X < +\infty$ *and that for every* $\varepsilon > 0$ *condition* (3.3) *is satisfied. Then* $\Sigma_{\mu,\nu}$ *have the form given in* (P1), (P2), (P3) *or* (P4) *and in the latter two cases* $k \leq \dim X + 1$.

Proof. Since $\dim X < +\infty$, it follows that $\kappa_{\rm ic} = -\infty$ and $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}} = \Sigma_{\mu,\nu}$. In particular, the description of $\Sigma_{\mu,\nu}^{\kappa_{\rm ic}}$ given in Theorem 3.1 is actually a description of the whole spectrum $\Sigma_{\mu,\nu}$. Moreover, proceeding as in the proof of Claim 1 with $d = \dim X$, we conclude that $\Sigma_{\mu,\nu}$ may have at most $d+1$ different connected components proving that $k \leq \dim X + 1$ and that the only possible options for $\Sigma_{\mu,\nu}$ are (P1)-(P4). $\square$

3.2. **Some extra properties of $\Sigma_{\mu,\nu}$.** In this section we describe some properties of $\Sigma_{\mu,\nu}$ under some extra conditions.

Lemma 3.9. *Suppose there exists* $K, \chi > 0$ *such that*

$$\|\mathcal{A}(m,n)\| \leq K\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{\chi} \quad \text{for } m \geq n. \tag{3.18}$$

Then $\Sigma_{\mu,\nu} \subset (-\infty, \chi]$ *and* $S_\gamma(n) = X$ *for every* $\gamma \geq \chi$ *and* $n \in \mathbb{Z}$.

Proof. Given $\varepsilon > 0$, condition (3.18) implies that

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^{\chi+\varepsilon} \mathcal{A}(m,n) \right\| \leq K\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-\varepsilon} \quad \text{for } m \geq n.$$

Consequently, $\chi + \varepsilon \in \rho_{\mu,\nu}$ (with $P_n = \mathrm{Id}$ for every $n \in \mathbb{Z}$) and $\Sigma_{\mu,\nu} \subset (-\infty, \chi]$. The second claim follows directly from (3.18) and the definition of $S_\gamma(n)$. $\square$

Similarly, we have the following.

Lemma 3.10. *Suppose that* A_n, $n \in \mathbb{Z}$, *is invertible and there exists* $K, \chi > 0$ *such*

$$\|\mathcal{A}(m,n)\| \leq K\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{\chi} \quad \text{for } n \geq m. \tag{3.19}$$

Then $\Sigma_{\mu,\nu} \subset [-\chi, +\infty)$ *and* $U_\gamma(n) = X$ *for every* $\gamma \leq -\chi$ *and* $n \in \mathbb{Z}$.

If there exists $K, \chi > 0$ such that (3.18) and (3.19) are satisfied we say that (2.1) has (μ,ν)*-bounded growth*.

Corollary 3.11. *Suppose that* A_n *is invertible for each* $n \in \mathbb{Z}$ *and that there exist* $K, \chi > 0$ *such that* (3.18) *and* (3.19) *hold. Then,* $\Sigma_{\mu,\nu}$ *is bounded and non-empty.*

Proof. The first claim follows directly from Lemmas 3.9 and 3.10. Let us prove the second one. For this purpose, given $n \in \mathbb{Z}$, let us consider

$$c = \inf\{\gamma \in \rho_{\mu,\nu} : \ S_\gamma(n) = X\}.$$

Observe that Lemmas 3.9 and 3.10 imply that $-\chi \leq c \leq \chi$. For the sake of contradiction, suppose $c \in \rho_{\mu,\nu}$. Then, if $S_c(n) = X$, it follows by Lemma 3.5 that $S_{c-\varepsilon}(n) = X$ for some small ε which contradicts the definition of c. On the other hand, if $S_c(n) \neq X$, then by Lemma 3.5 we have that $S_{c+\varepsilon}(n) \neq X$ for some small ε which again contradicts the definition of c. Thus, $c \in \Sigma_{\mu,\nu}$ and $\Sigma_{\mu,\nu} \neq \emptyset$. $\square$

4. Examples

We now present a series of examples showing that all the possibilities (P1)-(P6) given in Theorem 3.1 can actually appear as the (μ,ν)-spectrum of (2.1) even in the case when each A_n is a compact operator. Throughout this section, we assume that (3.3) is satisfied. Moreover, recall that we denote by $\mathcal{A}_\gamma(m,n)$ the evolution operator associated with (2.9) for $\gamma\in\mathbb{R}$.

Example 4.1. Let us consider $A_n\colon\mathbb{R}\to\mathbb{R}$ given by $A_n\equiv 0$ for every $n\in\mathbb{Z}$. It is easy to see that, in this case, (2.9) admits a (μ,ν)-dichotomy with $P_n=\mathrm{Id}$ for every $n\in\mathbb{Z}$ and every $\gamma\in\mathbb{R}$. Consequently, $\Sigma_{\mu,\nu}=\emptyset$ for this sequence of maps and possibility (P1) actually does occur.

Example 4.2. Let $(\alpha_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\lim_{n\to-\infty}\alpha_n=-\infty$ and $\lim_{n\to+\infty}\alpha_n=+\infty$ and consider $A_n\colon\mathbb{R}\to\mathbb{R}$, $n\in\mathbb{Z}$, defined by

$$A_nx=\frac{\mu_{n+1}^{\alpha_{n+1}}}{\mu_n^{\alpha_n}}x.$$

Then,

$$\mathcal{A}(m,n)x=\frac{\mu_m^{\alpha_m}}{\mu_n^{\alpha_n}}x.$$

In particular, given $\gamma\in\mathbb{R}$ we have that

$$\mathcal{A}_\gamma(m,n)x=\frac{\mu_m^{\alpha_m-\gamma}}{\mu_n^{\alpha_n-\gamma}}x. \tag{4.1}$$

Now, if (2.9) admits a (μ,ν)-dichotomy, we have two possibilities: either $P_n=\mathrm{Id}$ for every $n\in\mathbb{Z}$ or $P_n=0$ for every $n\in\mathbb{Z}$. We will show that neither of them can happen for any $\gamma\in\mathbb{R}$. Indeed, suppose (2.9) admits a (μ,ν)-dichotomy with $P_n=\mathrm{Id}$ for every $n\in\mathbb{Z}$. Then, by (2.6), it follows that for every $n\in\mathbb{Z}$,

$$\lim_{m\to+\infty}\mathcal{A}_\gamma(m,n)=0.$$

On the other hand, recalling that $\lim_{m\to+\infty}\mu_m=+\infty$ and $\lim_{m\to+\infty}\alpha_m=+\infty$, by (4.1) we get that $\lim_{m\to+\infty}\mathcal{A}_\gamma(m,n)=+\infty$. Thus, (2.9) does not admit a (μ,ν)-dichotomy with $P_n=\mathrm{Id}$ for every $n\in\mathbb{Z}$. Similarly, suppose (2.9) admits a (μ,ν)-dichotomy with $P_n=0$ for every $n\in\mathbb{Z}$. Then, by (2.7), it follows that for every $n\in\mathbb{Z}$,

$$\lim_{m\to-\infty}\mathcal{A}_\gamma(m,n)=0.$$

On the other hand, recalling that $\lim_{m\to-\infty}\mu_m=0$ and $\lim_{m\to-\infty}\alpha_m=-\infty$, by (4.1) we get that $\lim_{m\to-\infty}\mathcal{A}_\gamma(m,n)=+\infty$ and (2.9) does not admit a (μ,ν)-dichotomy with $P_n=0$ for every $n\in\mathbb{Z}$. Consequently, $\gamma\in\Sigma_{\mu,\nu}$. Thus, since $\gamma\in\mathbb{R}$ was arbitrary, we get that $\Sigma_{\mu,\nu}=\mathbb{R}$ and possibility (P2) also does occur.

The objective of the next three examples is to build sequences of operators acting on $\mathbb{R}$ for which $\Sigma_{\mu,\nu}=[a,b]$ for $a\le b$, $\Sigma_{\mu,\nu}=[a,+\infty)$ and $\Sigma_{\mu,\nu}=(-\infty,b]$, respectively. This will then be used to build examples having the (μ,ν)-spectrum as in (P3), (P4), (P5) and (P6).

Example 4.3. We start with the case $\Sigma_{\mu,\nu}=[a,b]$ for $a\le b$. For this purpose, observe that, since $(\mu_n)_{n\in\mathbb{Z}}$ is strictly increasing and

$$\lim_{n\to-\infty}\mu_n=0 \quad\text{and}\quad \lim_{n\to+\infty}\mu_n=+\infty,$$

there exists $n_0 \in \mathbb{Z}$ such that $\mu_n < 1$ for every $n < n_0$ and $\mu_n \geq 1$ for every $n \geq n_0$. Then, given $a \leq b$, let us consider $A_n^{a,b} \colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{Z}$, defined as

$$A_n^{a,b}x = \begin{cases} \left(\frac{\mu_{n+1}}{\mu_n}\right)^b \frac{\nu_n}{\nu_{n+1}}x & \text{if } n \geq n_0 \\ \left(\frac{\mu_{n+1}}{\mu_n}\right)^a \frac{\nu_n}{\nu_{n+1}}x & \text{if } n < n_0. \end{cases}$$

Then, for $m \geq n$,

$$\mathcal{A}(m,n)x = \begin{cases} \left(\frac{\mu_m}{\mu_n}\right)^b \frac{\nu_n}{\nu_m}x & \text{if } m,n \geq n_0 \\ \mu_{n_0}^{a-b}\frac{\mu_m^b}{\mu_n^a}\frac{\nu_n}{\nu_m}x & \text{if } n < n_0 \leq m \\ \left(\frac{\mu_m}{\mu_n}\right)^a \frac{\nu_n}{\nu_m}x & \text{if } m,n \leq n_0. \end{cases}$$

Consequently, since $a \leq b$ and $\nu_m \geq 1$ for every $m \in \mathbb{Z}$, there exists $D > 0$ such that

$$\mathcal{A}(m,n) \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^b$$

for every $m \geq n$ (in order to verify that this inequality holds for $m \geq n_0 > n$, recall that $\mu_n < 1$ for $n < n_0$). Thus, for $\gamma > b$ we have that

$$\mathcal{A}_\gamma(m,n) \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-(\gamma-b)}$$

for every $m \geq n$ and (2.9) admits a (μ,ν)-dichotomy with $P_n = \mathrm{Id}$. In particular, $\gamma \notin \Sigma_{\mu,\nu}$.

Similarly, for $m < n$,

$$\mathcal{A}(m,n)x = \begin{cases} \left(\frac{\mu_m}{\mu_n}\right)^b \frac{\nu_n}{\nu_m}x & \text{if } m,n \geq n_0 \\ \mu_{n_0}^{b-a}\frac{\mu_m^a}{\mu_n^b}\frac{\nu_n}{\nu_m}x & \text{if } m < n_0 \leq n \\ \left(\frac{\mu_m}{\mu_n}\right)^a \frac{\nu_n}{\nu_m}x & \text{if } m,n \leq n_0. \end{cases}$$

Thus, since $a \leq b$ and $\nu_m \geq 1$ for every $m \in \mathbb{Z}$, there exists $D > 0$ such that

$$\mathcal{A}(m,n) \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^a$$

for every $m < n$ (in order to verify that this inequality holds for $m < n_0 \leq n$, recall that $\mu_n \geq 1$ for $n \geq n_0$). Consequently, for $\gamma < a$ we have that

$$\mathcal{A}_\gamma(m,n) \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-(\gamma-a)} = D\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-|\gamma-a|}$$

and (2.9) admits a (μ,ν)-dichotomy with $P_n = 0$. In particular, $\gamma \notin \Sigma_{\mu,\nu}$.

Now, let us consider $\gamma \in [a,b]$. Then, we observe that

$$\mathcal{A}_\gamma(m,n)x = \begin{cases} \left(\frac{\mu_m}{\mu_n}\right)^{b-\gamma} \frac{\nu_n}{\nu_m}x & \text{if } m,n \geq n_0 \\ \left(\frac{\mu_m}{\mu_n}\right)^{a-\gamma} \frac{\nu_n}{\nu_m}x & \text{if } m,n \leq n_0. \end{cases}$$

Thus, analyzing the expression for $\mathcal{A}_\gamma(m,n)$ in the case when $m,n \geq n_0$ we conclude that (2.9) does not admit a (μ,ν)-dichotomy with $P_n = \mathrm{Id}$ for every $n \in \mathbb{Z}$. Similarly, analyzing the expression for $\mathcal{A}_\gamma(m,n)$ in the case when $m,n \leq n_0$ we conclude that (2.9) does not admit a (μ,ν)-dichotomy with $P_n = 0$ for every $n \in \mathbb{Z}$. Consequently, $\gamma \in \Sigma_{\mu,\nu}$. Combining our previous observations, we conclude that $\Sigma_{\mu,\nu} = [a,b]$.

Example 4.4. We will now modify our construction in Example 4.3 in order to obtain a sequence of operators acting on $\mathbb{R}$ for which $\Sigma_{\mu,\nu} = [a, +\infty)$ for $a \in \mathbb{R}$. So, let n_0 be as in the aforementioned example, $a \in \mathbb{R}$ and $(\alpha_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\alpha_n > a$ for every $n \in \mathbb{Z}$ and $\lim_{n\to+\infty} \alpha_n = +\infty$ and consider $A_n^a : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{Z}$, defined as

$$A_n^a x = \begin{cases} \frac{\mu_{n+1}^{\alpha_{n+1}}}{\mu_n^{\alpha_n}} \frac{\nu_n}{\nu_{n+1}} x & \text{if } n \geq n_0 \\ \left(\frac{\mu_{n+1}}{\mu_n}\right)^a \frac{\nu_n}{\nu_{n+1}} x & \text{if } n < n_0. \end{cases}$$

Then, for $m < n$,

$$\mathcal{A}(m,n)x = \begin{cases} \frac{\mu_m^{\alpha_m}}{\mu_n^{\alpha_n}} \frac{\nu_n}{\nu_m} x & \text{if } m, n \geq n_0 \\ \mu_{n_0}^{\alpha_{n_0}-a} \frac{\mu_m^a}{\mu_n^{\alpha_n}} \frac{\nu_n}{\nu_m} x & \text{if } m < n_0 \leq n \\ \left(\frac{\mu_m}{\mu_n}\right)^a \frac{\nu_n}{\nu_m} x & \text{if } m, n \leq n_0. \end{cases}$$

Thus, since $\alpha_m > a$ and $\nu_m \geq 1$ for every $m \in \mathbb{Z}$, there exists $D > 0$ such that

$$\mathcal{A}(m,n) \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^a$$

for every $m < n$ (recall that $\mu_n \geq 1 > \mu_m$ for $n \geq n_0 > m$). Consequently, for $\gamma < a$ we have that

$$\mathcal{A}_\gamma(m,n) \leq D\nu_n \left(\frac{\mu_m}{\mu_n}\right)^{-(\gamma-a)} = D\nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-|\gamma-a|}$$

and (2.9) admits a (μ,ν)-dichotomy with $P_n = 0$. In particular, $\gamma \notin \Sigma_{\mu,\nu}$.

Let us now consider $\gamma \geq a$. Then, analyzing the expression for $\mathcal{A}_\gamma(m,n)$ in the case when $m, n \leq n_0$, we conclude that (2.9) does not admit a (μ,ν)-dichotomy with $P_n = 0$ for every $n \in \mathbb{Z}$. On the other hand, observing that for $m \geq n$,

$$\mathcal{A}(m,n)x = \begin{cases} \frac{\mu_m^{\alpha_m}}{\mu_n^{\alpha_n}} \frac{\nu_n}{\nu_m} x & \text{if } m, n \geq n_0 \\ \mu_{n_0}^{a-\alpha_{n_0}} \frac{\mu_m^{\alpha_m}}{\mu_n^a} \frac{\nu_n}{\nu_m} x & \text{if } n < n_0 \leq m \\ \left(\frac{\mu_m}{\mu_n}\right)^a \frac{\nu_n}{\nu_m} x & \text{if } m, n \leq n_0 \end{cases}$$

and analyzing the expression for $\mathcal{A}_\gamma(m,n)$ in the case when $m, n \geq n_0$, recalling (3.3) and that $\alpha_m > \gamma$ for every m sufficiently large, we conclude that $\mathcal{A}_\gamma(m,n) \to +\infty$ as $m \to +\infty$. In particular, (2.9) can not admit a (μ,ν)-dichotomy with $P_n = \mathrm{Id}$ for every $n \in \mathbb{Z}$. Thus, $\gamma \in \Sigma_{\mu,\nu}$ and, consequently, $\Sigma_{\mu,\nu} = [a, +\infty)$.

Example 4.5. Given n_0 as in Example 4.3 and $b \in \mathbb{R}$, let $(\alpha_n)_{n\in\mathbb{Z}}$ be an increasing sequence such that $\alpha_n < b$ for every $n \in \mathbb{Z}$ and $\lim_{n\to-\infty} \alpha_n = -\infty$ and consider $A_n^b : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{Z}$, defined as

$$A_n^b x = \begin{cases} \left(\frac{\mu_{n+1}}{\mu_n}\right)^b \frac{\nu_n}{\nu_{n+1}} x & \text{if } n \geq n_0 \\ \frac{\mu_{n+1}^{\alpha_{n+1}}}{\mu_n^{\alpha_n}} \frac{\nu_n}{\nu_{n+1}} x & \text{if } n < n_0. \end{cases}$$

Then, proceeding as in Example 4.4 we can show that if $\gamma > b$ then (2.9) admits a (μ,ν)-exponential dichotomy with $P_n = \mathrm{Id}$. On the other hand, if $\gamma \leq b$ then we can proceed again as above to show that $\gamma \in \Sigma_{\mu,\nu}$. Thus, $\Sigma_{\mu,\nu} = (-\infty, b]$.

From now on we fix the Banach space

$$X = \left\{ (w_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=0}^{+\infty} w_n^2 < +\infty \right\}$$

endowed with the norm

$$\|(w_n)_{n\in\mathbb{N}}\|_2 = \left(\sum_{n=0}^{+\infty} w_n^2\right)^{\frac{1}{2}}.$$

As promised above, we will use Examples 4.3, 4.4 and 4.5 to build examples of systems acting on X with spectrum as in (P3)-(P6).

Example 4.6. Given numbers

$$a_k \le b_k < a_{k-1} \le b_{k-1} < \ldots < a_1 \le b_1, \tag{4.2}$$

let us consider the compact operator $A_n\colon X \to X$, $n \in \mathbb{Z}$, given by

$$A_n x = (A_n^{a_1,b_1} x_1, A_n^{a_2,b_2} x_2, \ldots, A_n^{a_k,b_k} x_k, 0, 0, \ldots)$$

for $x = (x_1, x_2, \ldots) \in X$ where $A_n^{a_j,b_j}$ is as in Example 4.3 for $j = 1, 2, \ldots, k$. We will now observe that

$$\Sigma_{\mu,\nu} = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_k, b_k].$$

Let $\gamma \in \mathbb{R} \setminus \bigcup_{j=1}^k [a_j, b_j]$. If $\gamma > b_1$, let us consider $P_n = \mathrm{Id}$ for every $n \in \mathbb{Z}$; if $\gamma \in (b_j, a_{j-1})$, let us consider

$$P_n x = (0, \ldots, 0, x_j, x_{j+1}, \ldots)$$

for $x = (x_1, x_2, \ldots)$ and $n \in \mathbb{Z}$; and finally, if $\gamma < a_k$ let us consider

$$P_n x = (0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)$$

for $x = (x_1, x_2, \ldots)$ and $n \in \mathbb{Z}$. Then, (2.9) admits a (μ, ν)-dichotomy with these projections since by Example 4.3 we have that $\left(\left(\frac{\mu_n}{\mu_{n+1}}\right)^{\gamma} A_n^{a_j,b_j}\right)_{n\in\mathbb{Z}}$ admits either a (μ, ν)-contraction or a (μ, ν)-expansion for each $j = 1, 2, \ldots, k$. Therefore, $\Sigma_{\mu,\nu} \subset \bigcup_{j=1}^k [a_j, b_j]$. Let us now prove that the reverse inclusion also holds.

We will proceed by contradiction. Suppose there exist $j \in \{1, \ldots, k\}$ and $\gamma \in [a_j, b_j]$ such that (2.9) admits a (μ, ν)-dichotomy with projections $(P_n)_{n\in\mathbb{Z}}$. Let us consider the one-dimensional subspace $X_j \subset X$ which consists of all the elements of X of the form $(0, \ldots 0, x_j, 0, \ldots)$ with $x_j \in \mathbb{R}$. Then, $A_n X_j = X_j$ for every $n \in \mathbb{Z}$ and, consequently, either $X_j \subset \operatorname{Im} P_n$ for every $n \in \mathbb{Z}$ or $X_j \subset \operatorname{Ker} P_n$ for every $n \in \mathbb{Z}$. In particular, the action of $\left(\left(\frac{\mu_n}{\mu_{n+1}}\right)^{\gamma} A_n\right)_{n\in\mathbb{Z}}$ restricted to X_j, which is basically given by $\left(\left(\frac{\mu_n}{\mu_{n+1}}\right)^{\gamma} A_n^{a_j,b_j}\right)_{n\in\mathbb{Z}}$, must be either a (μ, ν)-contraction or a (μ, ν)-expansion. On the other hand, we have observed in Example 4.3 that this is not the case, giving us a contradiction. Therefore, combining the previous observations we conclude that $\Sigma_{\mu,\nu} = \bigcup_{j=1}^k [a_j, b_j]$.

Example 4.7. Given numbers as in (4.2), let us consider the compact operator $A_n\colon X \to X$, $n \in \mathbb{Z}$, given by

$$A_n x = (A_n^{a_1} x_1, A_n^{a_2,b_2} x_2, \ldots, A_n^{a_k,b_k} x_k, 0, 0, \ldots)$$

for $x = (x_1, x_2, \ldots) \in X$ where A^{a_1} is as in Example 4.4 and $A_n^{a_j,b_j}$ is as in Example 4.3 for $j = 2, \ldots, k$. By proceeding as in Example 4.6 we conclude that

$$\Sigma_{\mu,\nu} = [a_k, b_k] \cup \ldots \cup [a_2, b_2] \cup [a_1, +\infty).$$

Now, combining Examples 4.6 and 4.7 and recalling that in both cases $\kappa_{\mathrm{ic}} = -\infty$ we conclude that possibility (P3) does occur.

Example 4.8. Let us again consider numbers as in (4.2) and let $A_n : X \to X$, $n \in \mathbb{Z}$, be given by

$$A_n x = (A_n^{a_1} x_1, A_n^{a_2,b_2} x_2, \dots, A_n^{a_{k-1},b_{k-1}} x_{k-1}, A_n^{b_k} x_k, 0, 0, \dots)$$

for $x = (x_1, x_2, \dots) \in X$ where A^{a_1} is as in Example 4.4, $A_n^{a_j,b_j}$ is as in Example 4.3 for $j = 2, \dots, k-1$ and A^{b_k} is as in Example 4.5. Proceeding again as in Example 4.6 we conclude that

$$\Sigma_{\mu,\nu} = (-\infty, b_k] \cup [a_{k-1}, b_{k-1}] \cup \ldots \cup [a_2, b_2] \cup [a_1, +\infty).$$

Then, since each A_n, $n \in \mathbb{Z}$, is a compact operator, we have that $\kappa_{\text{ic}} = -\infty$ and, consequently, possibility (P4) also does occur.

Example 4.9. For our next example, consider numbers

$$\ldots < a_3 \le b_3 < a_2 \le b_2 < a_1 \le b_1 \tag{4.3}$$

with $\lim\limits_{j \to +\infty} a_j = -\infty$ and let $A_n : X \to X$, $n \in \mathbb{Z}$, be given by

$$A_n x = (A_n^{a_1,b_1} x_1, A_n^{a_2,b_2} x_2, A_n^{a_3,b_3} x_3, \dots)$$

for $x = (x_1, x_2, \dots) \in X$ where $A_n^{a_j,b_j}$ is as in Example 4.3 for $j = 1, 2, 3, \dots$. Then, each operator A_n, $n \in \mathbb{Z}$, is compact. Indeed, given the canonical basis $(e_j)_{j \in \mathbb{N}}$ of X, we have that $A_n e_j = \lambda_j e_j$ for every j with $\lim_{j \to +\infty} \lambda_j = 0$. Thus, since X is a separable Hilbert space, it follows by [18, Proposition 4.6] that A_n is compact as claimed. In particular, $\Sigma_{\mu,\nu}^{\kappa_{\text{ic}}} = \Sigma_{\mu,\nu}$. Now, proceeding as in Example 4.6 we get that $\Sigma_{\mu,\nu} = \bigcup_{j=1}^{+\infty} [a_j, b_j]$. Similarly, changing $A_n^{a_1,b_1}$ by $A_n^{a_1}$ in the definition of A_n above where $A_n^{a_1}$ is as in Example 4.4, we get an example of dynamics having $\Sigma_{\mu,\nu} = \bigcup_{j=2}^{+\infty} [a_j, b_j] \cup [a_1, +\infty)$. This shows that possibility (P5) also occurs.

Example 4.10. For our final example, consider numbers as in (4.3) with $\lim\limits_{j \to +\infty} a_j =: b_\infty \in \mathbb{R}$. Then, let $A_n : X \to X$, $n \in \mathbb{Z}$, be given by

$$A_n x = (A_n^{b_\infty} x_1, A_n^1 x_2, A_n^2 x_3, A_n^3 x_4, \dots)$$

for $x = (x_1, x_2, \dots) \in X$ where $A_n^{b_\infty}$ is as in Example 4.5 and $A_n^j : \mathbb{R} \to \mathbb{R}$

$$A_n^j y = \begin{cases} \left(\frac{\mu_{n+1}}{\mu_n} \right)^{b_j} \frac{1}{\alpha_{n,j}} y & \text{if } n \ge n_0 \\ \left(\frac{\mu_{n+1}}{\mu_n} \right)^{a_j} \frac{1}{\alpha_{n,j}} y & \text{if } n < n_0 \end{cases}$$

for $j = 1, 2, 3, \dots$, $y \in \mathbb{R}$ and n_0 as in Example 4.3 where $\alpha_{n,j} \ge 1$ are numbers such that $\alpha_{n,j} \to +\infty$ as $j \to +\infty$ and $\prod_{n \in \mathbb{Z}} \alpha_{n,j} < +\infty$ for every j (we may take, for instance, $\alpha_{n,j} = 1 + j e^{-|n|}$). In particular, $A_n^j \to 0$ as $j \to +\infty$ and, by [18, Proposition 4.6], A_n is compact and $\Sigma_{\mu,\nu}^{\kappa_{\text{ic}}} = \Sigma_{\mu,\nu}$. In fact, this last fact is the reason why we have used A_n^j instead of $A_n^{a_j,b_j}$ to construct the operator A_n.

Then, proceeding as in Example 4.3 we obtain that the (μ, ν)-spectrum of $(A_n^j)_{n \in \mathbb{Z}}$ is $[a_j, b_j]$ for $j = 1, 2, 3, \dots$. Finally, using this observation and Example 4.5 and proceeding as in Example 4.6, we conclude that the (μ, ν)-spectrum of $(A_n)_{n \in \mathbb{Z}}$ is equal to $(-\infty, b_\infty] \cup \bigcup_{j=1}^{\infty} [a_j, b_j]$. Similarly, changing A_n^1 by $A_n^{a_1}$ in the definition of A_n where $A_n^{a_1}$ is as in Example 4.4 we get an example of dynamics with (μ, ν)-spectrum equal to $(-\infty, b_\infty] \cup \bigcup_{j=2}^{\infty} [a_j, b_j] \cup [a_1, +\infty)$. This shows that possibility (P6) also does occur.

5. Variations of the (μ, ν)-dichotomy spectrum

In this section we consider two slight variations of the notion of (μ, ν)-dichotomy considered in the previous sections and present a classification of the dichotomy spectrum associated with these notions.

5.1. **Generalized (μ,ν)-dichotomy spectrum.** We say that (2.1) admits a *generalized (μ,ν)-dichotomy* if conditions (2.4) and (2.5) are satisfied and, moreover, there exist $D,\lambda>0$ and $\theta\geq 0$ such that

$$\|\mathcal{A}(m,n)P_n\|\leq D\nu_n^\theta\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda}\qquad \text{for } m\geq n$$

and

$$\|\mathcal{A}(m,n)(\mathrm{Id}-P_n)\|\leq D\nu_n^\theta\left(\frac{\mu_n}{\mu_m}\right)^{-\lambda}\qquad \text{for } m\leq n$$

where $\mathcal{A}(m,n)$ is as in (2.8) for $m\leq n$. Observe that in the case when $\theta=1$ we recover the notion of (μ,ν)-dichotomy.

We define the *generalized (μ,ν)-dichotomy spectrum* of (2.1) as the set of all numbers $\gamma\in\mathbb{R}$ for which the system (2.9) *does not* admit a generalized (μ,ν)-dichotomy and denote this set by $\Sigma^g_{\mu,\nu}$. The set $\rho^g_{\mu,\nu}:=\mathbb{R}\setminus\Sigma_{\mu,\nu}$ is called the *generalized (μ,ν)-resolvent* set of (2.1). It is easy to see that $\Sigma^g_{\mu,\nu}\subset\Sigma_{\mu,\nu}$ and, in general, we may have $\Sigma^g_{\mu,\nu}\neq\Sigma_{\mu,\nu}$ as we show in the next example.

Example 5.1. For $n\in\mathbb{Z}$, let $\mu_n=e^n$ and

$$\nu_n=\begin{cases}1 & \text{if } n=0 \text{ or } |n|=2^k \text{ for some } k\in\mathbb{N}\\ |n| & \text{otherwise.}\end{cases}$$

Now, let $A_n\colon\mathbb{R}\to\mathbb{R}$ be given by

$$A_nx=\frac{\mu_n}{\mu_{n+1}}\frac{\nu_n^2}{\nu_{n+1}^2}x\ \text{ for every } x\in\mathbb{R}.$$

Thus,

$$\mathcal{A}(m,n)=\frac{\mu_n}{\mu_m}\frac{\nu_n^2}{\nu_m^2}\ \text{ and }\ \mathcal{A}_\gamma(m,n)=\left(\frac{\mu_n}{\mu_m}\right)^{1+\gamma}\frac{\nu_n^2}{\nu_m^2}$$

for every $\gamma\in\mathbb{R}$ and $m,n\in\mathbb{Z}$. Consequently,

$$\mathcal{A}_\gamma(m,n)=\begin{cases}\nu_n^2e^{-(1+\gamma)(m-n)} & \text{if } m=0 \text{ or } |m|=2^k \text{ for some } k\in\mathbb{N}\\ \frac{\nu_n^2}{m^2}e^{-(1+\gamma)(m-n)} & \text{otherwise.}\end{cases}\tag{5.1}$$

Then, it is easy to see that, for every $\gamma\geq 0$, the system (2.9) admits a generalized (μ,ν)-dichotomy with parameters $D=1$, $\lambda=1+\gamma$, $\theta=2$ and $P_n=\mathrm{Id}$ for every $n\in\mathbb{Z}$. In particular, $[0,+\infty)\cap\Sigma^g_{\mu,\nu}=\emptyset$. On the other hand, (2.9) does not admit a (μ,ν)-dichotomy for any $\gamma\geq 0$. Indeed, suppose (2.9) admits a (μ,ν)-dichotomy for some $\gamma\geq 0$. Then, by the second line in (5.1) we have that $P_n=\mathrm{Id}$ for every $n\in\mathbb{Z}$. Consequently, there exist $D>0$ and $\lambda>0$ such that, for every $k\in\mathbb{N}$, $m=2^k$ and $m\geq n$,

$$\nu_n^2e^{-(1+\gamma)(m-n)}\leq D\nu_ne^{-\lambda(m-n)}$$

which is equivalent to

$$\nu_n\leq De^{(1+\gamma-\lambda)(m-n)}.$$

In particular, taking $n=2^k-1$ we get from the previous inequality that

$$2^k-1=\nu_{2^k-1}\leq De^{(1+\gamma-\lambda)}$$

which is a contradiction since $k\in\mathbb{N}$ is arbitrary. Therefore, $[0,+\infty)\subset\Sigma_{\mu,\nu}$ and $\Sigma_{\mu,\nu}\neq\Sigma^g_{\mu,\nu}$.

Remark 5.2. We observe that in [31], Silva considered a similar notion of spectrum for continuous time invertible dynamics acting on a finite-dimensional space in the particular case where ν is given in terms of μ and, moreover, the exponent θ in the definition of the dichotomy has some restrictions based on λ.

In the next result, we describe the structure of the set $\Sigma^{g,\kappa_{\rm ic}}_{\mu,\nu} := \Sigma^g_{\mu,\nu} \cap (\kappa_{\rm ic}, +\infty)$.

Theorem 5.3. *Suppose $\kappa_{ic} < 0$ and that for every $\varepsilon > 0$ condition (3.3) is satisfied. Then $\Sigma^{g,\kappa_{ic}}_{\mu,\nu}$ has one of the forms given in (P1)-(P6). Moreover, conclusions (3.4), (3.5), (3.6), (3.7) and (3.8) are also satisfied. Furthermore, all the forms (P1)-(P6) do appear as the generalized (μ,ν)-dichotomy spectrum of compact operators.*

Proof. The proof of this result follows the same lines, mutatis mutandis, as the proof of Theorem 3.1. Moreover, it is easy to adapt the examples presented in Section 4 to the present context. ☐

5.2. **Nonuniform μ-dichotomy spectrum.** We say that (2.1) admits a *nonuniform μ-dichotomy* if conditions (2.4) and (2.5) are satisfied and, moreover, there exists $\lambda > 0$ so that for each $\theta > 0$ there exists $D = D(\theta) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le D\mu_n^{\operatorname{sgn}(n)\theta} \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \qquad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)(\operatorname{Id} - P_n)\| \le D\mu_n^{\operatorname{sgn}(n)\theta} \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda} \qquad \text{for } m \le n$$

where $\mathcal{A}(m,n)$ is as in (2.8) for $m \le n$ and $\operatorname{sgn}(n)$ denotes the "sign" of n.

Similarly to what we did above, we define the *nonuniform μ-dichotomy spectrum* of (2.1) as the set of all numbers $\gamma \in \mathbb{R}$ for which the system (2.9) *does not* admit a nonuniform μ-dichotomy and denote this set by Σ^N_μ. The set $\rho^N_\mu := \mathbb{R} \setminus \Sigma^N_\mu$ is called the *nonuniform μ-resolvent* set of (2.1). Observe that letting $n_0 \in \mathbb{Z}$ be such that $\mu_n < 1$ for $n < n_0$ and $\mu_n \ge 1$ for $n \ge n_0$ as in Example 4.3 and considering

$$\nu_n = \begin{cases} \mu_n & \text{if } n \ge n_0 \\ \mu_n^{-1} & \text{if } n < n_0, \end{cases}$$

we have that

$$\Sigma^g_{\mu,\nu} \subset \Sigma_{\mu,\nu} \subset \Sigma^N_\mu.$$

Moreover, in general, we have that $\Sigma_{\mu,\nu} \neq \Sigma^N_\mu$.

Example 5.4. For $n \in \mathbb{Z}$, let $\mu_n = e^n$ and $\nu_n = e^{|n|}$. Moreover, let $A_n \colon \mathbb{R} \to \mathbb{R}$ be given by

$$A_n x = e^{-\frac{5}{2} + \frac{(n+1)\cos(n+1) - n\cos(n)}{2}} x \quad \text{for every } x \in \mathbb{R}.$$

Then,

$$\mathcal{A}(m,n) = e^{-\frac{5}{2}(m-n) + \frac{m\cos(m) - n\cos(n)}{2}}$$

for every $m, n \in \mathbb{Z}$. Now, it follows from [6, Example 2] with $c = 5/2$ and $b = 1/2$ that $\Sigma^N_\mu = \mathbb{R}$. On the other hand, we have that

$$\mathcal{A}(m,n) \le e^{-\frac{5}{2}(m-n) + \frac{|m|+|n|}{2}} \le e^{-2(m-n)+|n|} = \nu_n \left(\frac{\mu_n}{\mu_m}\right)^{-2}$$

for every $m \ge n$. In particular, $0 \notin \Sigma_{\mu,\nu}$ and $\Sigma_{\mu,\nu} \neq \Sigma^N_\mu$.

In the next result, we describe the structure of the set $\Sigma^{N,\kappa_{\rm ic}}_{\mu} := \Sigma^N_\mu \cap (\kappa_{\rm ic}, +\infty)$.

Theorem 5.5. *Suppose $\kappa_{ic} < 0$. Then $\Sigma^{N,\kappa_{ic}}_\mu$ has one of the forms given in (P1)-(P6). Moreover, conclusions (3.4), (3.5), (3.6), (3.7) and (3.8) are also satisfied. Furthermore, all the forms (P1)-(P6) do appear as the nonuniform μ-dichotomy spectrum of compact operators.*

Proof. The proof of this result again follows the same lines, mutatis mutandis, as the proof of Theorem 3.1 and, moreover, the examples presented in Section 4 can be easily adapted to the present context. ☐

Remark 5.6. We observe that by taking $\mu_n = e^n$ for every $n \in \mathbb{Z}$, our notion of nonuniform μ-dichotomy spectrum coincides with the spectrum introduced and studied in [6]. In particular, in this case, our Theorem 5.5 recovers [6, Theorem 4]. Furthermore, in [5] the authors consider a notion similar to our concept of nonuniform μ-dichotomy spectrum whose main difference is that they work with finite-dimensional invertible systems and use the notion of *strong* nonuniform μ-dichotomy to define the spectrum instead of just that of nonuniform μ-dichotomy as we do.

Remark 5.7. Versions of the results from Section 3.2 can also be obtained for the generalized (μ, ν)-dichotomy spectrum and the nonuniform μ-dichotomy spectrum.

6. Cohomology and normal forms

Many of the applications obtained for the classical Sacker-Sell spectrum can be adapted to obtain versions in terms of the spectra introduced above. In this section, we present two such adaptations and show how the nonuniform μ-dichotomy spectrum can be used to "reduce" (2.1) into a system which has a block-diagonal form and comment on how one can use this spectrum to obtain normal forms of certain nonautonomous systems. We follow the ideas developed in [8, 27, 29].

6.1. **Cohomology.** We say that a sequence $(T_n)_{n\in\mathbb{Z}}$ of invertible linear maps in $\mathcal{B}(X)$ is *μ-tempered* if for every $\varepsilon > 0$,

$$\lim_{n\to\pm\infty} \frac{\|T_n\|}{\mu_n^{\operatorname{sgn}(n)\varepsilon}} = 0 \quad \text{and} \quad \lim_{n\to\pm\infty} \frac{\|T_n^{-1}\|}{\mu_n^{\operatorname{sgn}(n)\varepsilon}} = 0 \tag{6.1}$$

where as above $\operatorname{sgn}(n)$ denotes the "sign" of n. Two sequences $(A_n)_{n\in\mathbb{Z}}$ and $(B_n)_{n\in\mathbb{Z}}$ are said to be *μ-cohomologous* if there exists a sequence of μ-tempered linear maps $(T_n)_{n\in\mathbb{Z}}$ such that

$$B_n = T_{n+1}^{-1} A_n T_n. \tag{6.2}$$

Proposition 6.1. *If $(A_n)_{n\in\mathbb{Z}}$ and $(B_n)_{n\in\mathbb{Z}}$ are μ-cohomologous then they have the same nonuniform μ-dichotomy spectrum.*

In what follows, we are going to denote by $\mathcal{B}(m,n)$ the evolution operator associated to $(B_n)_{n\in\mathbb{Z}}$ which is obtained by changing A_k by B_k, $k \in \mathbb{Z}$, in (2.2).

Proof of Proposition 6.1. Let $\gamma \in \rho_\mu^N(A)$ where $\rho_\mu^N(A)$ denotes the nonuniform μ-resolvent of $(A_n)_{n\in\mathbb{Z}}$. Then (2.9) admits a nonuniform μ-dichotomy. That is, there exists a family of projections $(P_n)_{n\in\mathbb{Z}}$ such that (2.4) and (2.5) hold and, moreover, there exists a constant $\lambda > 0$ such that for every $\theta > 0$ there exists $D = D(\theta) > 0$ satisfying

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{A}(m,n)P_n \right\| \leq D\mu_n^{\operatorname{sgn}(n)\theta} \left(\frac{\mu_m}{\mu_n}\right)^{-\lambda} \quad \text{for } m \geq n \tag{6.3}$$

and

$$\left\| \left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{A}(m,n)(\operatorname{Id} - P_n) \right\| \leq D\mu_n^{\operatorname{sgn}(n)\theta} \left(\frac{\mu_n}{\mu_m}\right)^{-\lambda} \quad \text{for } m \leq n. \tag{6.4}$$

Then, considering the family of projections $(\tilde{P}_n)_{n\in\mathbb{Z}}$ given by $\tilde{P}_n = T_n^{-1} P_n T_n$, conditions (2.4) and (6.2) imply that $B_n \tilde{P}_n = \tilde{P}_{n+1} B_n$. Moreover, conditions (2.5) and (6.2) imply that

$$B_n|_{\operatorname{Ker} \tilde{P}_n} : \operatorname{Ker} \tilde{P}_n \to \operatorname{Ker} \tilde{P}_{n+1}$$

is invertible.

Now, using (6.1) we have that for every $\varepsilon \in (0,\lambda/2)$ there exists $C_\varepsilon > 0$ such that $\|T_n\| \le C_\varepsilon \mu_n^{\mathrm{sgn}(n)\varepsilon}$ and $\|T_n^{-1}\| \le C_\varepsilon \mu_n^{\mathrm{sgn}(n)\varepsilon}$ for every $n \in \mathbb{Z}$. Thus, using (6.2) and (6.3), for $m \ge n$ we have the following possibilities:

- $m \ge 0$ and $n \le 0$:

$$\begin{aligned}\left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{B}(m,n)\tilde{P}_n\right\| &= \left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma T_m^{-1}\mathcal{A}(m,n)P_nT_n\right\|\\ &\le \|T_n\|\|T_m^{-1}\|\left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{A}(m,n)P_n\right\|\\ &\le D\mu_n^{\mathrm{sgn}(n)\theta}C_\varepsilon\mu_n^{\mathrm{sgn}(n)\varepsilon}C_\varepsilon\mu_m^{\mathrm{sgn}(m)\varepsilon}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda}\\ &= DC_\varepsilon^2\mu_n^{\mathrm{sgn}(n)\theta}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda+\varepsilon}\\ &\le \hat{D}\mu_n^{\mathrm{sgn}(n)(\theta+2\varepsilon)}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda+\varepsilon}\end{aligned}$$

for some $\hat{D} > 0$ where here we have used that $\lim_{n\to-\infty}\mu_n = 0$ and, in particular, $\mu_n^{\mathrm{sgn}(n)\theta} \le \mu_n^{\mathrm{sgn}(n)(\theta+2\varepsilon)}$ for n small enough;

- $m, n \ge 0$:

$$\begin{aligned}\left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{B}(m,n)\tilde{P}_n\right\| &\le DC_\varepsilon^2\mu_n^{\mathrm{sgn}(n)\theta}\mu_n^{\mathrm{sgn}(n)\varepsilon}\mu_m^{\mathrm{sgn}(m)\varepsilon}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda}\\ &= DC_\varepsilon^2\mu_n^{\theta}\mu_n^{2\varepsilon}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda+\varepsilon}\\ &\le DC_\varepsilon^2\mu_n^{\mathrm{sgn}(n)(\theta+2\varepsilon)}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda+\varepsilon}\end{aligned}$$

- $m, n \le 0$:

$$\begin{aligned}\left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{B}(m,n)\tilde{P}_n\right\| &\le DC_\varepsilon^2\mu_n^{\mathrm{sgn}(n)\theta}\mu_n^{\mathrm{sgn}(n)\varepsilon}\mu_m^{\mathrm{sgn}(m)\varepsilon}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda}\\ &= DC_\varepsilon^2\mu_n^{-\theta}\mu_n^{-2\varepsilon}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda-\varepsilon}\\ &\le DC_\varepsilon^2\mu_n^{\mathrm{sgn}(n)(\theta+2\varepsilon)}\left(\frac{\mu_m}{\mu_n}\right)^{-\lambda-\varepsilon}\end{aligned}$$

Thus, combining the previous observations and taking $\tilde{\lambda} = \lambda/2 > 0$ we get that for every $\tilde{\theta} > 0$ there exists $\tilde{D} = \tilde{D}(\theta) > 0$ such that

$$\left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{B}(m,n)\tilde{P}_n\right\| \le \tilde{D}\mu_n^{\mathrm{sgn}(n)\tilde{\theta}}\left(\frac{\mu_m}{\mu_n}\right)^{-\tilde{\lambda}} \quad \text{for } m \ge n.$$

Similarly we can prove that

$$\left\|\left(\frac{\mu_n}{\mu_m}\right)^\gamma \mathcal{B}(m,n)(\mathrm{Id}-\tilde{P}_n)\right\| \le \tilde{D}\mu_n^{\mathrm{sgn}(n)\tilde{\theta}}\left(\frac{\mu_n}{\mu_m}\right)^{-\tilde{\lambda}} \quad \text{for } m \le n.$$

Consequently, $\gamma \in \rho_\mu^N(B)$ showing that $\rho_\mu^N(A) \subset \rho_\mu^N(B)$. By changing the roles of $(A_n)_{n\in\mathbb{Z}}$ and $(B_n)_{n\in\mathbb{Z}}$ in the previous argument we conclude that $\rho_\mu^N(B) \subset \rho_\mu^N(A)$ completing the proof of the proposition. □

6.2. **Block-diagonalization.** Our next objective is to show that a sequence $(A_n)_{n\in\mathbb{Z}}$ of bounded linear operators satisfying $\kappa_{\text{ic}} < 0$ is μ-cohomologous to one in block form. For this purpose, given integers $k_1 < k_2 < \ldots < k_s$, by Theorem 5.5 we may write

$$X = F(n) \oplus \bigoplus_{j=1}^{s} E_{k_j}(n) \tag{6.5}$$

for every $n \in \mathbb{Z}$ where

$$F(n) = S_{c_{k_s+1}}(n) \oplus U_{c_{k_1-1}}(n) \oplus \bigoplus_{j\in S} E_j(n)$$

and $S = (\mathbb{N} \cap [k_1, k_s]) \setminus \{k_1, k_2, \ldots, k_s\}$. Denote by $n_j := \dim(E_{k_j}(n))$, which is finite and independent of $n \in \mathbb{Z}$, and consider $d = n_1 + n_2 + \ldots + n_s$.

Theorem 6.2. *Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of bounded linear operators such that $\kappa_{ic} < 0$ and with decomposition as in (6.5). Then, $(A_n)_{n\in\mathbb{Z}}$ is μ-cohomologous to a sequence $(B_n)_{n\in\mathbb{Z}}$ of linear operators from $\mathbb{R}^d \times F(n)$ to $\mathbb{R}^d \times F(n+1)$ with*

$$B_n = diag(B_n^1, B_n^2, \ldots, B_n^{s+1})$$

where $B^j = (B_n^j)_{n\in\mathbb{Z}}$ is a sequence of invertible $n_j \times n_j$ matrices associated with the spaces $E_{k_j}(n)$ with

$$\Sigma_\mu^N(B^j) = [a_{k_j}, b_{k_j}], \quad for\ j = 1, \ldots, s,$$

and

$$B_n^{s+1} = A_n|_{F(n)} \colon F(n) \to F(n+1).$$

Proof. The proof of this result can be obtained by making minor adjustments in the proof of [8, Theorem 19] and, therefore, we refrain from writing full details. □

6.3. **Normal forms.** For $x \in X$, let us write $x = (x^1, \ldots, x^s, x^{s+1})$ with $x^j \in \mathbb{R}^{n_j}$ for $j = 1, \ldots, s$ and $x^{s+1} \in F(n)$. Observe that x^{s+1} depends on n but, since this dependence is clear from the context, we will abuse notation and not write it. Given a vector $q = (q_1, \ldots, q_s) \in \mathbb{N}^s$, we define $|q| = q_1 + \ldots + q_s$ and $\partial^q f_n = \partial_{x^1}^{q_1} \cdots \partial_{x^s}^{q_s} f_n$.

Theorem 6.3. *Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of bounded linear operators such that $\kappa_{ic} < 0$ with decomposition as in (6.5) and suppose that $f_n \colon X \to X$ are maps of class C^r with $f_n(0) = 0$ and $d_0 f_n = 0$ for every $n \in \mathbb{Z}$. Moreover, suppose that for each $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that*

$$\|d_0^q f_n\| \leq K \mu_n^{sgn(n)\varepsilon} \tag{6.6}$$

for $2 \leq |q| \leq r$ and every $n \in \mathbb{Z}$. Then, there exists a μ-tempered sequence of linear operators $(T_n)_{n\in\mathbb{Z}}$ such that if $(x_n)_{n\in\mathbb{Z}}$ satisfies

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \in \mathbb{Z},$$

then $y_n = T_n^{-1} x_n$ satisfies

$$y_{n+1} = B_n y_n + g_n(y_n), \quad n \in \mathbb{Z} \tag{6.7}$$

for some linear operators B_n as in Theorem 6.2 and some maps $g_n \colon X \to X$ of class C^r with $g_n(0) = 0$ and $d_0 g_n = 0$ for every $n \in \mathbb{Z}$ having the property that for each $\varepsilon > 0$ there exists $L = L(\varepsilon) > 0$ such that

$$\|d_0^q g_n\| \leq L \mu_n^{sgn(n)\varepsilon} \tag{6.8}$$

for $2 \leq |q| \leq r$ and every $n \in \mathbb{Z}$.

Proof. By Theorem 6.2, there exists a μ-tempered sequence of linear operators $(T_n)_{n\in\mathbb{Z}}$ and linear operators $(B_n)_{n\in\mathbb{Z}}$ of the form given in the cited theorem such that $B_n = T_{n+1}^{-1}A_nT_n$. Thus, making $y_n = T_n^{-1}x_n$ and considering $g_n = T_{n+1}^{-1}\circ f_n \circ T_n$ we have that

$$\begin{aligned} y_{n+1} = T_{n+1}^{-1}x_{n+1} &= T_{n+1}^{-1}A_nx_n + T_{n+1}^{-1}f_n(x_n) \\ &= T_{n+1}^{-1}A_nT_ny_n + T_{n+1}^{-1}f_n(T_ny_n) \\ &= B_ny_n + g_n(y_n). \end{aligned}$$

Finally, condition (6.8) follows directly from (6.1) and (6.6). □

Given $j \in \{1, 2, \ldots, s\}$ and $q \in \mathbb{N}^s$ with $|q| \geq 2$, we say that the pair (j, q) is a *resonance* of order $|q|$ if

$$[a_{k_j}, b_{k_j}] \bigcap \left[\sum_{i=1}^{s} q_i a_{k_i}, \sum_{i=1}^{s} q_i b_{k_i}\right] \neq \emptyset \tag{6.9}$$

where $k_1 < \ldots < k_s$ are as in Section 6.2.

Now, let $g_n \colon X \to X$ be maps of class C^r satisfying $g_n(0) = 0$ and $d_0g_n = 0$ for every $n \in \mathbb{Z}$. We have

$$g_n(x) = \sum_{q\in\mathbb{N}^s, 2\leq|q|\leq r} \frac{1}{q!} d_0^q g_n x^q + o(\|x\|^r). \tag{6.10}$$

Then, writing

$$g_n = (g_n^1, g_n^2, \ldots, g_n^s, g_n^{s+1}),$$

(recall that we are identifying X with $\mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_s} \times F(n)$), we say that the component $(1/q!)d_0^q g_m^j x^q$ in (6.10) is *resonant* if the pair (j, q) is a resonance.

Theorem 6.4. *Let $(B_n)_{n\in\mathbb{Z}}$ be a sequence of linear operators as in Theorem 6.2 and $g_n \colon X \to X$ be maps of class C^r with $g_n(0) = 0$ and $d_0g_n = 0$ for every $n \in \mathbb{Z}$. Moreover, suppose that for each $\varepsilon > 0$ there exists $L = L(\varepsilon) > 0$ satisfying (6.8). Then, there exist polynomials $h_n \colon \mathbb{R}^d \to \mathbb{R}^d$ with $h_n(0) = 0$ and $d_0h_n = 0$ for every $n \in \mathbb{Z}$ such that, if $(y_n)_{n\in\mathbb{Z}}$ satisfies (6.7), then making*

$$y_n^j = z_n^j + h_n^j(z_n^1, \ldots, z_n^s)$$

for $j \in \{1, \ldots, s\}$ and $y_n^{s+1} = z_n^{s+1}$ for every $n \in \mathbb{Z}$, we get that

$$z_{n+1} = B_nz_n + \bar{g}_n(z_n), \quad n \in \mathbb{Z},$$

for some maps $\bar{g}_n \colon X \to X$ of class C^r with $\bar{g}_n(0) = 0$, $d_0\bar{g}_n = 0$ and $d_0^q\bar{g}_n^j = 0$ for all $n \in \mathbb{Z}$, $j \in \{1, \ldots, s\}$ and $q \in \mathbb{N}^s$ with $2 \leq |q| \leq r$ such that (j, q) is not a resonance, where $\bar{g}_n^j$ are the components of $\bar{g}_n$.

Proof. The proof of this result is analogous to the proof of [8, Theorem 21] and, therefore, once again we refrain from writing full details. □

Remark 6.5. We observe that reducibility results and normal forms associated with the μ-dichotomy spectrum studied in [31] were recently obtained in [12, 31] in the context of continuous time dynamics acting on a finite-dimensional space.

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