SHADOWING FOR NONAUTONOMOUS AND NONLINEAR DYNAMICS WITH IMPULSES

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Abstract. For a large class of nonautonomous semilinear impulsive differential equations, we formulate sufficient conditions under which in a vicinity of each approximate solution, we can construct an exact solution. An important feature of our result is that it is applicable to situations when the linear part is not hyperbolic. In addition, we establish analogous result in the case of discrete time.

1. Introduction

It is well known that, in general, it is either complicated or even impossible to explicitly solve a given differential equation. Nowadays, a variety of numerical schemes for approximating solutions (over large intervals of time) of many classes of differential equations are available. Naturally, any numerical scheme will in general result with only an approximate solution of a given differential equation. The information given by an approximate solution will be useful only in situations when in a vicinity of this approximate solution, there exists an exact solution of our differential equation. The differential equations exhibiting this property are said to have the *shadowing property* (see [15, 16]). We note that this notion includes the notion of the Hyers-Ulam stability (see [7]) as a particular case. Indeed, the latter notion requires a precise estimate for the deviation of an approximate solution from an exact solution in terms of the error in the approximation.

In their recent paper [4], the first two authors investigated the shadowing property of semilinear differential equations of the form

$$x' = A(t)x + f(t,x) \quad t \geq 0, \tag{1.1}$$

where A is a continuous map taking values in the space of all bounded linear operators on some Banach space X and $f\colon [0,+\infty) \times X \to X$ is continuous. It is proved in [4, Corollary 3.3] that if $x' = A(t)x$ admits the so-called (μ,ν)-dichotomy and if $f(t,\cdot)$ is Lipschitz with a suitable Lipschitz constant for each $t \geq 0$, then (1.1) has the shadowing property. In fact, it was showed (see [4, Theorem 3.1]) that under some weaker assumptions for the linear part $x' = A(t)x$, (1.1) exhibits

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a certain milder form of the shadowing property. We stress that the arguments in [4] build on the works [2, 3], where the particular case when $x' = A(t)x$ admits a (generalized) exponential dichotomy was studied. We emphasize that related results have been obtained earlier (either for discrete or continuous time) under assumptions that the linear part A is constant or periodic and that the nonlinear part f vanishes [5, 6, 8, 9, 12, 20, 21].

The main objective of the present paper is to study the shadowing property for a class of semilinear *impulsive* differential equations of the form

$$\begin{aligned} x' &= A(t)x + f(t,x), \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &:= x(\tau_i+) - x(\tau_i-) = C_i x(\tau_i-) + p_i(x(\tau_i-)). \end{aligned} \tag{1.2}$$

Here, $(\tau_n)_{n\in\mathbb{Z}}$ is an arbitrary sequence in $\mathbb{R}$ such that

$$\lim_{n\to-\infty} \tau_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} \tau_n = \infty.$$

Moreover, $A(t)$ and C_i are bounded linear operators acting on a Banach space X, $f\colon \mathbb{R}\times X \to X$, $p_i\colon X \to X$ are nonlinear terms, while $x(t+)$ and $x(t-)$ denote the limit from the right and from the left of x in t, respectively. In Theorem 1, we formulate sufficient conditions under which (1.2) exhibits the shadowing property. The main novelty with respect to results in [4] (besides considering equations with impulses) is that our Theorem 1 requires no assumptions related to the asymptotic behaviour for the linear part of (1.2) (see Subsection 2.4). In the particular case, when the linear part of (1.2) admits an exponential dichotomy, Theorem 1 gives sufficient conditions under which (1.2) exhibits the Hyers-Ulam stability property (see Theorem 2). While there are several important works related to the Hyers-Ulam stability for different classes of impulsive equations (see for example [11, 18, 19] and references therein), we stress that even our Theorem 2 is a completely new result.

In Section 3, we prove the version of Theorem 1 in the case of discrete time. We note that our arguments are inspired by those developed in [1, 2, 3, 4, 10], which in turn are inspired by the analytic proofs of the shadowing lemma in the context of smooth dynamics [13, 14].

2. A continuous time case

2.1. **Preliminaries.** Consider a Banach space $(X, |\cdot|)$ and the space of all bounded linear operators on X, denoted by $\mathcal{B}(X)$. We denote the operator norm on $\mathcal{B}(X)$ by $\|\cdot\|$. Let $(\tau_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence such that

$$\lim_{n\to-\infty} \tau_n = -\infty \quad \text{and} \quad \lim_{n\to\infty} \tau_n = \infty.$$

Moreover, let $A\colon \mathbb{R} \to \mathcal{B}(X)$ be locally integrable. Furthermore, let $(C_n)_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{B}(X)$ such that $\mathrm{Id} + C_n$ is an invertible operator for each $n \in \mathbb{Z}$. We consider a linear impulsive differential equation given by,

$$\begin{aligned} x' &= A(t)x, \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &:= x(\tau_i+) - x(\tau_i-) = C_i x(\tau_i-), \end{aligned} \tag{2.1}$$

where for any $h\colon \mathbb{R} \to X$ by $h(x+)$ and $h(x-)$ we denote the limit of h in x from right and left, respectively.

Let $T(t,s)$ denote the evolution operator associated to (2.1). Assume that $P\colon \mathbb{R} \to \mathcal{B}(X)$ is a measurable map and set

$$\mathcal{G}(t,s) = \begin{cases} T(t,s)P(s) & t \geq s; \\ -T(t,s)(\mathrm{Id} - P(s)) & t \leq s. \end{cases}$$

Finally, let $f\colon \mathbb{R} \times X \to X$ and $p_n\colon X \to X$, $n \in \mathbb{Z}$, be such that:

- $t \mapsto f(t,x)$ is locally integrable for each $x \in X$;
- there exists a Borel measurable $c\colon \mathbb{R} \to (0,+\infty)$ such that

$$|f(t,x) - f(t,y)| \leq c(t)|x-y| \quad \text{for } t \in \mathbb{R} \text{ and } x,y \in X \tag{2.2}$$

  and

$$|p_n(x) - p_n(y)| \leq d_n|x-y| \quad \text{for } n \in \mathbb{Z} \text{ and } x,y \in X, \tag{2.3}$$

  for some sequence $(d_n)_{n\in\mathbb{Z}} \subset (0,+\infty)$ satisfying

$$\sup_{n\in\mathbb{Z}} (d_n \|(\mathrm{Id} + C_n)^{-1}\|) < 1. \tag{2.4}$$

Now, we consider the associated impulsive semilinear differential equation given by

$$\begin{aligned} x' &= A(t)x + f(t,x), \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i-) + p_i(x(\tau_i-)). \end{aligned} \tag{2.5}$$

The above conditions ensure the existence and the uniqueness (with the prescribed initial condition) of global right-continuous solutions of (2.5). We note that (2.4) is needed to ensure continuability of solutions of (2.5) in the negative direction. We refer to [17] for details.

2.2. **A shadowing type result.** We are now in a position to formulate our first result.

Theorem 1. *Assume that*

$$q := \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{\infty} c(s)\|\mathcal{G}(t,s)\|\, ds + \sum_{i\in\mathbb{Z}} d_i \|\mathcal{G}(t,\tau_i)\| \right) < 1. \tag{2.6}$$

Furthermore, let $\varepsilon\colon \mathbb{R} \to (0,+\infty)$ be a measurable function and $(\varepsilon_i)_{i\in\mathbb{Z}}$ a sequence in $(0,+\infty)$ such that

$$L := \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \varepsilon(s)\|\mathcal{G}(t,s)\|\, ds + \sum_{i\in\mathbb{Z}} \varepsilon_i \|\mathcal{G}(t,\tau_i)\| \right) < +\infty. \tag{2.7}$$

Then, for each map $z\colon \mathbb{R}\to X$ differentiable on each interval (τ_n,τ_{n+1}), $n\in\mathbb{Z}$, satisfying

$$|z'(t)-A(t)z(t)-f(t,z(t))|\le \varepsilon(t) \qquad \text{for } t\neq \tau_i, \tag{2.8}$$

and

$$|\Delta z|_{t=\tau_i}-C_iz(\tau_i-)-p_i(z(\tau_i-))|\le \varepsilon_i, \qquad \text{for } i\in\mathbb{Z}, \tag{2.9}$$

there exists a solution $x\colon \mathbb{R}\to X$ of (2.5) such that

$$|x(t)-z(t)|\le \frac{L}{1-q}, \qquad \text{for every } t\in\mathbb{R}. \tag{2.10}$$

Moreover, if (2.1) admits no non-trivial bounded solution then the solution $x\colon\mathbb{R}\to X$ given above is unique.

Proof. Let $\mathcal{Y}$ denote the space of all $x\colon\mathbb{R}\to X$ that are continuous on each $[\tau_i,\tau_{i+1})$ for each $i\in\mathbb{Z}$ having only discontinuities of the first kind in τ_i and such that

$$\|x\|_\infty := \sup_{t\in\mathbb{R}}|x(t)|<+\infty.$$

Then, $(\mathcal{Y},\|\cdot\|_\infty)$ is a Banach space. For $y\in\mathcal{Y}$, we define $\mathcal{T}y$ by

$$\begin{aligned}(\mathcal{T}y)(t)&=\int_{-\infty}^{\infty}\mathcal{G}(t,s)(A(s)z(s)+f(s,y(s)+z(s))-z'(s))\,ds\\&\quad+\sum_{i\in\mathbb{Z}}\mathcal{G}(t,\tau_i)(C_iz(\tau_i-)-z(\tau_i+)+z(\tau_i-)+p_i(z(\tau_i-)+y(\tau_i-))),\end{aligned}\tag{2.11}$$

for $t\in\mathbb{R}$. We first claim that $\mathcal{T}$ is well-defined. Take $y\in\mathcal{Y}$. It follows from (2.2) and (2.8) that

$$\begin{aligned}&|A(s)z(s)+f(s,y(s)+z(s))-z'(s)|\\&\le |A(s)z(s)+f(s,z(s))-z'(s)|+|f(s,y(s)+z(s))-f(s,z(s))|\\&\le \varepsilon(s)+c(s)|y(s)|,\end{aligned}$$

for each $s\neq\tau_i$, $i\in\mathbb{Z}$. Hence, by (2.6) and (2.7) we have that

$$\begin{aligned}&\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{\infty}\mathcal{G}(t,s)(A(s)z(s)+f(s,y(s)+z(s))-z'(s))\,ds\right|\\&\le \sup_{t\in\mathbb{R}}\left(\int_{-\infty}^{\infty}\varepsilon(s)\|\mathcal{G}(t,s)\|\,ds\right)\\&\quad+\|y\|_\infty\cdot\sup_{t\in\mathbb{R}}\left(\int_{-\infty}^{\infty}c(s)\|\mathcal{G}(t,s)\|\,ds\right)\\&\le L+q\|y\|_\infty.\end{aligned}\tag{2.12}$$

Similarly, it follows from (2.3) and (2.9) that

$$\begin{aligned}
&|C_i z(\tau_i-) - z(\tau_i+) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-))| \\
&\le |C_i z(\tau_i-) - z(\tau_i+) + z(\tau_i-) + p_i(z(\tau_i-))| \\
&\quad + |p_i(z(\tau_i-) + y(\tau_i-)) - p_i(z(\tau_i-))| \\
&\le \varepsilon_i + d_i|y(\tau_i-)| \\
&\le \varepsilon_i + d_i\|y\|_\infty,
\end{aligned}$$

for each $i \in \mathbb{Z}$. Consequently, by (2.6) and (2.7) we have that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left| \sum_{i\in\mathbb{Z}} \mathcal{G}(t,\tau_i)(C_i z(\tau_i-) - z(\tau_i+) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-))) \right| \\
&\le \sup_{t\in\mathbb{R}} \left(\sum_{i\in\mathbb{Z}} \varepsilon_i \|\mathcal{G}(t,\tau_i)\| \right) + \|y\|_\infty \cdot \sup_{t\in\mathbb{R}} \left(\sum_{i\in\mathbb{Z}} d_i \|\mathcal{G}(t,\tau_i)\| \right) \\
&\le L + q\|y\|_\infty.
\end{aligned}$$

The above estimate together with (2.12) implies that $\mathcal{T}y \in \mathcal{Y}$. Thus, $\mathcal{T}$ is well-defined. Moreover, by (2.7), (2.8) and (2.9), we have that

$$\|\mathcal{T}0\|_\infty \le L. \tag{2.13}$$

Take now arbitrary $y_1, y_2 \in \mathcal{Y}$. It follows from (2.2) and (2.3) that

$$\begin{aligned}
&|\mathcal{T}y_1(t) - \mathcal{T}y_2(t)| \\
&\le \int_{-\infty}^{\infty} |\mathcal{G}(t,s)\big(f(s, y_1(s) + z(s)) - f(s, y_2(s) + z(s))\big)|\, ds \\
&+ \sum_{i\in\mathbb{Z}} |\mathcal{G}(t,\tau_i)\big(p_i(z(\tau_i-) + y_1(\tau_i-)) - p_i(z(\tau_i-) + y_2(\tau_i-))\big)| \\
&\le \int_{-\infty}^{\infty} c(s)\|\mathcal{G}(t,s)\| \cdot |y_1(s) - y_2(s)|\, ds \\
&+ \sum_{i\in\mathbb{Z}} d_i \|\mathcal{G}(t,\tau_i)\| \cdot |y_1(\tau_i-) - y_2(\tau_i-)|,
\end{aligned}$$

for $t \in \mathbb{R}$. Therefore, using (2.6),

$$\|\mathcal{T}y_1 - \mathcal{T}y_2\|_\infty \le q\|y_1 - y_2\|_\infty, \quad \text{for } y_1, y_2 \in \mathcal{Y}. \tag{2.14}$$

Set $C := \frac{L}{1-q} > 0$, and

$$\mathcal{D} := \{y \in \mathcal{Y} : \|y\|_\infty \le C\}. \tag{2.15}$$

We claim that $\mathcal{T}(\mathcal{D}) \subset \mathcal{D}$. To see this, take an arbitrary $y \in \mathcal{D}$. Then,

$$\begin{aligned}
\|\mathcal{T}y\|_\infty &\le \|\mathcal{T}0\|_\infty + \|\mathcal{T}y - \mathcal{T}0\|_\infty \\
&\le L + q\|y\|_\infty \\
&\le L + qC \\
&= C,
\end{aligned}$$

and thus $\mathcal{T}y \in \mathcal{D}$. Therefore, $\mathcal{T}|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is a contraction map. In particular, it has a unique fixed point $y \in \mathcal{D}$ such that $\mathcal{T}y = y$. Hence, for $t \geq \tau$ we have that

$$
\begin{aligned}
&y(t) - T(t,\tau)y(\tau) \qquad (2.16)\\
&= \int_{-\infty}^{t} T(t,s)P(s)(A(s)z(s) + f(s,z(s)+y(s)) - z'(s))\,ds\\
&\quad - \int_{-\infty}^{\tau} T(t,s)P(s)(A(s)z(s) + f(s,z(s)+y(s)) - z'(s))\,ds\\
&\quad - \int_{t}^{\infty} T(t,s)(\mathrm{Id} - P(s))(A(s)z(s) + f(s,z(s)+y(s)) - z'(s))\,ds\\
&\quad + \int_{\tau}^{\infty} T(t,s)(\mathrm{Id} - P(s))(A(s)z(s) + f(s,z(s)+y(s)) - z'(s))\,ds\\
&\quad + \sum_{\tau_i \leq t} T(t,\tau_i)P(\tau_i)(C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)))\\
&\quad - \sum_{\tau_i \leq \tau} T(t,\tau_i)P(\tau_i)(C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)))\\
&\quad - \sum_{\tau_i > t} T(t,\tau_i)(\mathrm{Id} - P(\tau_i))(C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)))\\
&\quad + \sum_{\tau_i > \tau} T(t,\tau_i)(\mathrm{Id} - P(\tau_i))(C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)))\\
&= \int_{\tau}^{t} T(t,s)(A(s)z(s) + f(s,z(s)+y(s)) - z'(s))\,ds\\
&\quad + \sum_{\tau < \tau_i \leq t} T(t,\tau_i)(C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-))).
\end{aligned}
$$

Now, for $t \neq \tau_i$, $i \in \mathbb{Z}$, differentiating in both sides of the equality we get that

$$
\begin{aligned}
&y'(t) - A(t)T(t,\tau)y(\tau)\\
&= A(t)z(t) + f(t,z(t)+y(t)) - z'(t)\\
&\quad + A(t)\int_{\tau}^{t} T(t,s)(A(s)z(s) + f(s,z(s)+y(s)) - z'(s))\,ds\\
&\quad + A(t)\sum_{\tau < \tau_i \leq t} T(t,\tau_i)(C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)))\\
&= A(t)z(t) + f(t,z(t)+y(t)) - z'(t) + A(t)\left(y(t) - T(t,\tau)y(\tau)\right)
\end{aligned}
$$

which implies that

$$
y'(t) + z'(t) = A(t)y(t) + A(t)z(t) + f(t,z(t)+y(t)). \qquad (2.17)
$$

Similarly, given $i \in \mathbb{Z}$ and taking $t = \tau_i$ and $\tau \to \tau_i-$ in (2.16) we obtain that

$$\begin{aligned} &y(\tau_i) - C_i y(\tau_i-) - y(\tau_i-) \\ &= C_i z(\tau_i-) - z(\tau_i) + z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)), \end{aligned}$$

which implies that

$$\begin{aligned} &y(\tau_i) + z(\tau_i) - y(\tau_i-) - z(\tau_i-) \\ &= C_i y(\tau_i-) + C_i z(\tau_i-) + p_i(z(\tau_i-) + y(\tau_i-)). \end{aligned} \tag{2.18}$$

Thus, combining (2.17) and (2.18) we conclude that $x := y + z$ is a solution of (2.5). Finally, recalling that $y \in \mathcal{D}$, we have that (2.10) holds completing the proof of the first claim in the theorem.

Assume now that (2.1) admits no non-trivial bounded solution and suppose $\tilde{x} \colon \mathbb{R} \to X$ is a solution of (2.5) such that

$$|\tilde{x}(t) - z(t)| \le \frac{L}{1-q}, \quad \text{for every } t \in \mathbb{R}. \tag{2.19}$$

Set $\tilde{y} := \tilde{x} - z$. Observe that

$$\begin{aligned} (\mathcal{T}\tilde{y})(t) &= \int_{-\infty}^{\infty} \mathcal{G}(t,s)(A(s)z(s) + f(s, \tilde{y}(s) + z(s)) - z'(s))\, ds \\ &\quad + \sum_{i\in\mathbb{Z}} \mathcal{G}(t,\tau_i)(C_i z(\tau_i-) - z(\tau_i+) + z(\tau_i-) + p_i(z(\tau_i-) + \tilde{y}(\tau_i-))) \\ &= \int_{-\infty}^{\infty} \mathcal{G}(t,s)(A(s)\tilde{x}(s) - A(s)\tilde{y}(s) + f(s, \tilde{x}(s)) - z'(s))\, ds \\ &\quad + \sum_{i\in\mathbb{Z}} \mathcal{G}(t,\tau_i)(C_i \tilde{x}(\tau_i-) - C_i\tilde{y}(\tau_i-) - z(\tau_i+) + z(\tau_i-) + p_i(\tilde{x}(\tau_i-))) \\ &= \int_{-\infty}^{\infty} \mathcal{G}(t,s)(\tilde{x}'(s) - A(s)\tilde{y}(s) - z'(s))\, ds \\ &\quad + \sum_{i\in\mathbb{Z}} \mathcal{G}(t,\tau_i)(\Delta \tilde{x}|_{t=\tau_i} - C_i\tilde{y}(\tau_i-) - z(\tau_i+) + z(\tau_i-)) \\ &= \int_{-\infty}^{\infty} \mathcal{G}(t,s)(\tilde{y}'(s) - A(s)\tilde{y}(s))\, ds \\ &\quad + \sum_{i\in\mathbb{Z}} \mathcal{G}(t,\tau_i)(\Delta \tilde{y}|_{t=\tau_i} - C_i\tilde{y}(\tau_i-)). \end{aligned}$$

Now, given $t \ge \tau$ and proceeding as in (2.16), we obtain that

$$\begin{aligned} (\mathcal{T}\tilde{y})(t) - T(t,\tau)(\mathcal{T}\tilde{y})(\tau) &= \int_{\tau}^{t} T(t,s)(\tilde{y}'(s) - A(s)\tilde{y}(s))\, ds \\ &\quad + \sum_{\tau<\tau_i\le t} T(t,\tau_i)(\Delta \tilde{y}|_{t=\tau_i} - C_i\tilde{y}(\tau_i-))), \end{aligned}$$

which implies that

$$\begin{cases}(\mathcal{T}\tilde{y})'(t)-A(t)(\mathcal{T}\tilde{y})(t)=\tilde{y}'(t)-A(t)\tilde{y}(t), \quad t\neq\tau_i,\\ \Delta(\mathcal{T}\tilde{y})|_{t=\tau_i}-C_i(\mathcal{T}\tilde{y})(\tau_i-)=\Delta\tilde{y}|_{t=\tau_i}-C_i\tilde{y}(\tau_i-), \quad i\in\mathbb{Z}.\end{cases}$$

Therefore, $w(t):=(\mathcal{T}\tilde{y})(t)-\tilde{y}(t)$ is a solution of (2.1) such that $\|w\|_\infty<+\infty$. Consequently, by our hypothesis it follows that $w=0$ and thus $\mathcal{T}\tilde{y}=\tilde{y}$. Finally, observe that (2.19) implies that $\tilde{y}\in\mathcal{D}$. Therefore, $\tilde{x}-z$ is the unique fixed point of $\mathcal{T}$ in $\mathcal{D}$. The proof of the theorem is completed. ☐

Remark 1. We would like to point out that the uniqueness cannot hold, in general, if (2.1) has non-trivial bounded solutions. Indeed, assume that x_0 is a non-trivial bounded solution of (2.1). Then, $z=0$ can be regarded as an approximate solution of (2.1) which can be shadowed by itself and any scalar multiple of x_0.

The rest of this section is devoted to present some settings to which Theorem 1 may be applied.

2.3. **Exponential dichotomy.** We say that (2.1) admits an *exponential dichotomy* if:

(1) there exists a family of projections $P(t)$, $t\in\mathbb{R}$, such that for every $t,s\in\mathbb{R}$,
$$T(t,s)P(s)=P(t)T(t,s);$$
(2) there exist $C,\lambda>0$ such that
$$\|T(t,s)P(s)\|\le Ce^{-\lambda(t-s)} \quad \text{for } t\ge s,$$
and
$$\|T(t,s)(\mathrm{Id}-P(s))\|\le Ce^{-\lambda(s-t)} \quad \text{for } t\le s.$$

In particular, we have that

$$\|\mathcal{G}(t,s)\|\le Ce^{-\lambda|t-s|}, \quad \text{for } t,s\in\mathbb{R}. \tag{2.20}$$

Set

$$R(t)=\sum_{i\in\mathbb{Z}}e^{-\lambda|t-\tau_i|}, \quad t\in\mathbb{R},$$

and suppose there exist constants $c,d>0$ such that (2.2) and (2.3) are satisfied with $c(s)=c$ for every $s\in\mathbb{R}$ and $d_i=d$ for every $i\in\mathbb{Z}$.

The following result is an important consequence of Theorem 1. To the best of our knowledge, it has not been established earlier in the literature. However, we stress that in the absence of impulsive behaviour, it reduces to [3, Theorem 6].

Theorem 2. *Suppose that*

$$r:=\sup_{t\in\mathbb{R}}R(t)<+\infty \tag{2.21}$$

and let

$$L := r + \frac{2C}{\lambda}. \tag{2.22}$$

Moreover, assume that

$$q := \frac{2cC}{\lambda} + dCr < 1. \tag{2.23}$$

Then, for every $\varepsilon > 0$ *and any map* $z\colon \mathbb{R} \to X$ *differentiable on each interval* (τ_n, τ_{n+1}), $n \in \mathbb{Z}$, *satisfying*

$$|z'(t) - A(t)z(t) - f(t, z(t))| \le \varepsilon \qquad \text{for } t \neq \tau_i,$$

and

$$|\Delta z|_{t=\tau_i} - C_i z(\tau_i-) - p_i(z(\tau_i-))| \le \varepsilon, \qquad \text{for } i \in \mathbb{Z},$$

there exists a unique solution $x\colon \mathbb{R} \to X$ *of* (2.5) *such that*

$$|x(t) - z(t)| \le \frac{L\varepsilon}{1-q}, \quad \text{for every } t \in \mathbb{R}. \tag{2.24}$$

Proof. By using (2.20) and (2.23) we can easily see that

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{\infty} c\|\mathcal{G}(t,s)\|\, ds + \sum_{i\in\mathbb{Z}} d\|\mathcal{G}(t,\tau_i)\| \right) \\
&\le \sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{\infty} cCe^{-\lambda|t-s|}\, ds + \sum_{i\in\mathbb{Z}} dCe^{-\lambda|t-\tau_i|} \right) \\
&\le \frac{2cC}{\lambda} + dCr = q < 1.
\end{aligned}$$

Similarly, using (2.22),

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left(\int_{-\infty}^{\infty} \varepsilon\|\mathcal{G}(t,s)\|\, ds + \sum_{i\in\mathbb{Z}} \varepsilon\|\mathcal{G}(t,\tau_i)\| \right) \\
&\le \varepsilon \sup_{t\in\mathbb{R}} \left(C\int_{-\infty}^{\infty} e^{-\lambda|t-s|}\, ds + \sum_{i\in\mathbb{Z}} e^{-\lambda|t-\tau_i|} \right) \\
&\le \varepsilon \left(\frac{2C}{\lambda} + r \right) = L\varepsilon < +\infty.
\end{aligned}$$

In particular, (2.6) and (2.7) hold. Moreover, since (2.1) admits an exponential dichotomy, it does not admit any non-trivial bounded solution. Thus, the result follows readily from Theorem 1. □

Remark 2. Observe that (2.21) holds, for example, in the case when $\tau_n = n$ for every $n \in \mathbb{Z}$.

2.4. **Beyond exponential dichotomy.** We now present an example that shows that Theorem 1 can be applied to situations when (2.1) does not admit an exponential dichotomy. Let $(X, |\cdot|)$ and $(\tau_i)_{i\in\mathbb{Z}}$ be as in Subsection 2.1 and let $\varphi, \psi \colon \mathbb{R} \to (0, +\infty)$ be two C^1 functions. For every $t \in \mathbb{R}$ let us consider the maps $A(t)$ and $P(t)$ acting on $X \times X$ given by

$$A(t) = \begin{pmatrix} \left(\frac{\varphi'(t)}{\varphi(t)} - 1\right)\mathrm{Id} & 0 \\ 0 & \left(\frac{\psi'(t)}{\psi(t)} + 1\right)\mathrm{Id} \end{pmatrix} \quad \text{and} \quad P(t) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, let

$$C_i = \begin{pmatrix} 0 & 0 \\ 0 & -2\cdot \mathrm{Id} \end{pmatrix}, \quad i \in \mathbb{Z}.$$

Then, denoting by $\tau(t,s)$ the number of τ_i's that belong to the interval $(s,t]$ if $s<t$ or to $[t,s)$ if $t<s$ and setting $\tau(t,t)=0$, we have that

$$T(t,s) = \begin{pmatrix} \frac{\varphi(t)}{\varphi(s)} e^{-(t-s)}\mathrm{Id} & 0 \\ 0 & (-1)^{\tau(t,s)} \frac{\psi(t)}{\psi(s)} e^{(t-s)}\mathrm{Id} \end{pmatrix}.$$

Consequently,

$$\mathcal{G}(t,s) = \begin{cases} \begin{pmatrix} \frac{\varphi(t)}{\varphi(s)} e^{-(t-s)}\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} & t \ge s; \\ \begin{pmatrix} 0 & 0 \\ 0 & -(-1)^{\tau(t,s)} \frac{\psi(t)}{\psi(s)} e^{(t-s)}\mathrm{Id} \end{pmatrix} & t < s. \end{cases}$$

Now, let $\mu, \nu \colon \mathbb{R} \to (0, D]$, $D>0$, be any C^1 functions and consider $\varphi(t) = \mu(t)e^t$ and $\psi(t) = \nu(t)e^{-t}$. Then,

$$\mathcal{G}(t,s) = \begin{cases} \begin{pmatrix} \frac{\mu(t)}{\mu(s)}\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} & t \ge s; \\ \begin{pmatrix} 0 & 0 \\ 0 & -(-1)^{\tau(t,s)} \frac{\nu(t)}{\nu(s)}\mathrm{Id} \end{pmatrix} & t < s. \end{cases}$$

In particular,

$$\|\mathcal{G}(t,s)\| \le \begin{cases} \frac{D}{\mu(s)} & t \ge s; \\ \frac{D}{\nu(s)} & t < s. \end{cases}$$

Now, if $c, \varepsilon : \mathbb{R} \to (0, +\infty)$ and $(d_i)_{i\in\mathbb{Z}}$ and $(\varepsilon_i)_{i\in\mathbb{Z}}$ are such that

$$\int_{-\infty}^{\infty} \frac{c(s)}{\mu(s)}\, ds + \sum_{i\in\mathbb{Z}} \frac{d_i}{\mu(\tau_i)} < \frac{1}{2D} \text{ and } \int_{-\infty}^{\infty} \frac{c(s)}{\nu(s)}\, ds + \sum_{i\in\mathbb{Z}} \frac{d_i}{\nu(\tau_i)} < \frac{1}{2D}$$

and

$$\int_{-\infty}^{\infty} \frac{\varepsilon(s)}{\mu(s)}\, ds + \sum_{i\in\mathbb{Z}} \frac{\varepsilon_i}{\mu(\tau_i)} < +\infty \text{ and } \int_{-\infty}^{\infty} \frac{\varepsilon(s)}{\nu(s)}\, ds + \sum_{i\in\mathbb{Z}} \frac{\varepsilon_i}{\nu(\tau_i)} < +\infty,$$

then (2.6) and (2.7) are satisfied and Theorem 1 is applicable. Finally, observe that if we take μ and ν to be constant maps, (2.1) does not admit an exponential dichotomy.

3. A discrete time case

3.1. **Preliminaries.** Let $(X, |\cdot|)$ and $(\mathcal{B}(X), \|\cdot\|)$ be as in Section 2. Given a sequence $(A_n)_{n\in\mathbb{Z}}$ of bounded and invertible linear operators in $\mathcal{B}(X)$, we consider the associated *linear difference equation* given by

$$x_{n+1} = A_n x_n, \text{ for all } n \in \mathbb{Z}. \tag{3.1}$$

For $m, n \in \mathbb{Z}$, set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{for } m > n; \\ \mathrm{Id} & \text{for } m = n; \\ A_m^{-1}\dots A_{n-1}^{-1} & \text{for } m < n. \end{cases} \tag{3.2}$$

Let $(P_n)_{n\in\mathbb{Z}}$ be a sequence in $\mathcal{B}(X)$ and define

$$\mathcal{G}(m,n) = \begin{cases} \mathcal{A}(m,n)P_n & \text{for } m \geq n; \\ -\mathcal{A}(m,n)(\mathrm{Id} - P_n) & \text{for } m < n. \end{cases} \tag{3.3}$$

Finally, let $f_n\colon X \to X$, $n \in \mathbb{Z}$, be a sequence of maps such that, for each $n \in \mathbb{Z}$, there exist numbers $c_n > 0$ satisfying

$$|f_n(x) - f_n(y)| \leq c_n|x-y| \text{ for every } n \in \mathbb{Z} \text{ and } x, y \in X. \tag{3.4}$$

Associated to these choices, we consider the *semilinear difference equation* given by

$$x_{n+1} = A_n x_n + f_n(x_n) \text{ for all } n \in \mathbb{Z}. \tag{3.5}$$

Observe that (3.1) and (3.5) are the discrete time versions of (2.1) and (2.5). Indeed, in this setting we have an impulse at each moment of time n.

3.2. **A shadowing type result.**

Theorem 3. *Assume that*

$$q := \sup_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} c_{n-1} \|\mathcal{G}(m,n)\| \right) < 1 \tag{3.6}$$

and let $(\delta_n)_{n\in\mathbb{Z}}$ be a sequence in $(0, +\infty)$ such that

$$L := \sup_{m\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} \delta_{n-1} \|\mathcal{G}(m,n)\| \right) < +\infty. \tag{3.7}$$

Then, for each sequence $(z_n)_{n\in\mathbb{Z}} \subset X$ satisfying

$$|z_{n+1} - A_n z_n - f_n(z_n)| \leq \delta_n \quad \text{for all } n \in \mathbb{Z}, \tag{3.8}$$

there exists a sequence $(x_n)_{n\in\mathbb{Z}} \subset X$ *satisfying* (3.5) *such that*

$$|x_n - z_n| \le \frac{L}{1-q}, \quad \text{for every } n \in \mathbb{Z}. \tag{3.9}$$

Moreover, in the case when (3.1) *admits no non-trivial bounded solution we have that the sequence* $(x_n)_{n\in\mathbb{Z}}$ *given above is unique.*

Proof. We follow closely the arguments in the proof of Theorem 1. Let

$$\mathcal{Y} := \left\{ \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \subset X : \ \|\mathbf{y}\|_\infty := \sup_{n\in\mathbb{Z}} |y_n| < +\infty \right\}.$$

Then, $(\mathcal{Y}, \|\cdot\|_\infty)$ is a Banach space. For $\mathbf{y} \in \mathcal{Y}$, we set

$$(\mathcal{T}\mathbf{y})_n = \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k),$$

for all $n \in \mathbb{Z}$.

We start observing that $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ given by $\mathcal{T}\mathbf{y} = ((\mathcal{T}\mathbf{y})_n)_{n\in\mathbb{Z}}$ is well-defined. Indeed, given $\mathbf{y} \in \mathcal{Y}$, it follows from (3.4) and (3.8) that

$$\begin{aligned}
&|A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k| \\
&\le |A_{k-1}z_{k-1} + f_{k-1}(z_{k-1}) - z_k| + |f_{k-1}(y_{k-1} + z_{k-1}) - f_{k-1}(z_{k-1})| \\
&\le \delta_{k-1} + c_{k-1}|y_{k-1}|,
\end{aligned}$$

for every $k \in \mathbb{Z}$. Thus, by (3.6) and (3.7) we have that

$$\begin{aligned}
&\sup_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k) \right| \\
&\le \sup_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \delta_{k-1}\|\mathcal{G}(n,k)\| \right) + \|\mathbf{y}\|_\infty \cdot \sup_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} c_{k-1}\|\mathcal{G}(n,k)\| \right) \\
&\le L + q\|\mathbf{y}\|_\infty,
\end{aligned} \tag{3.10}$$

which implies that $\mathcal{T}\mathbf{y} \in \mathcal{Y}$. Thus, $\mathcal{T}$ is well-defined.

Now, given arbitrary $\mathbf{y}, \mathbf{w} \in \mathcal{Y}$, it follows from (3.4) that

$$\begin{aligned}
&|(\mathcal{T}\mathbf{y})_n - (\mathcal{T}\mathbf{w})_n| \\
&\le \sum_{k\in\mathbb{Z}} |\mathcal{G}(n,k)\big(f_{k-1}(y_{k-1} + z_{k-1}) - f_{k-1}(w_{k-1} + z_{k-1})\big)| \\
&\le \sum_{k\in\mathbb{Z}} c_{k-1}\|\mathcal{G}(n,k)\| \cdot |y_{k-1} - w_{k-1}|,
\end{aligned}$$

for every $n \in \mathbb{Z}$. Therefore, using (3.6),

$$\|\mathcal{T}\mathbf{y} - \mathcal{T}\mathbf{w}\|_\infty \le q\|\mathbf{y} - \mathbf{w}\|_\infty. \tag{3.11}$$

Let $C := \frac{L}{1-q} > 0$ and consider $\mathcal{D} := \{\mathbf{y} \in \mathcal{Y} : \|\mathbf{y}\|_\infty \le C\}$. We claim that $\mathcal{T}(\mathcal{D}) \subset \mathcal{D}$. Indeed, by (3.7) and (3.8) we have that

$$\|\mathcal{T}\mathbf{0}\|_\infty \le L.$$

Thus, given $\mathbf{y} \in \mathcal{D}$, inequality (3.11) implies that

$$\begin{aligned}\|\mathcal{T}\mathbf{y}\|_\infty &\le \|\mathcal{T}\mathbf{0}\|_\infty + \|\mathcal{T}\mathbf{y} - \mathcal{T}\mathbf{0}\|_\infty \\ &\le L + q\|\mathbf{y}\|_\infty \\ &\le L + qC \\ &= C,\end{aligned}$$

and thus $\mathcal{T}\mathbf{y} \in \mathcal{D}$. Therefore, $\mathcal{T}|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is a contraction map. Hence, it has a unique fixed point $\mathbf{y} \in \mathcal{D}$ such that $\mathcal{T}\mathbf{y} = \mathbf{y}$. Consequently, for any $n \in \mathbb{Z}$ we have that

$$\begin{aligned}&y_{n+1} - A_n y_n \\ &= \sum_{k\le n+1} \mathcal{A}(n+1,k)P_k(A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k) \\ &\quad - \sum_{k\ge n+2} \mathcal{A}(n+1,k)(\mathrm{Id} - P_k)(A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k) \\ &\quad - \sum_{k\le n} \mathcal{A}(n+1,k)P_k(A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k) \\ &\quad + \sum_{k\ge n+1} \mathcal{A}(n+1,k)(\mathrm{Id} - P_k)(A_{k-1}z_{k-1} + f_{k-1}(y_{k-1} + z_{k-1}) - z_k) \\ &= P_{n+1}(A_n z_n + f_n(y_n + z_n) - z_{n+1}) \\ &\quad + (\mathrm{Id} - P_{n+1})(A_n z_n + f_n(y_n + z_n) - z_{n+1}) \\ &= A_n z_n + f_n(y_n + z_n) - z_{n+1}.\end{aligned}$$

Hence,

$$y_{n+1} + z_{n+1} = A_n(y_n + z_n) + f_n(y_n + z_n), \quad n \in \mathbb{Z}.$$

Consequently, $\mathbf{x} := \mathbf{y} + \mathbf{z}$ is a solution of (3.5) and, moreover, since $\mathbf{y} \in \mathcal{D}$, it also satisfies (3.9) thus completing the proof of the first part of the theorem.

Assume now that (3.1) admits no non-trivial bounded solution and let $\tilde{\mathbf{x}} = (\tilde{x}_n)_{n\in\mathbb{Z}}$ be a sequence satisfying (3.5) and (3.9) and consider $\tilde{\mathbf{y}} = (\tilde{y}_n)_{n\in\mathbb{Z}}$ given by $\tilde{y}_n = \tilde{x}_n - z_n$. In particular, $\|\tilde{\mathbf{y}}\|_\infty \le L/(1-q)$.

Then,

$$\begin{aligned}(\mathcal{T}\tilde{\mathbf{y}})_n &= \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(A_{k-1}z_{k-1} + f_{k-1}(\tilde{y}_{k-1} + z_{k-1}) - z_k) \\ &= \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(A_{k-1}\tilde{x}_{k-1} - A_{k-1}\tilde{y}_{k-1} + f_{k-1}(\tilde{x}_{k-1}) - z_k) \\ &= \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(\tilde{x}_k - A_{k-1}\tilde{y}_{k-1} - z_k) \\ &= \sum_{k\in\mathbb{Z}} \mathcal{G}(n,k)(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &= \sum_{k\le n} \mathcal{A}(n,k)P_k(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &\quad - \sum_{k\ge n+1} \mathcal{A}(n,k)(\mathrm{Id} - P_k)(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}),\end{aligned}$$

which implies that

$$\begin{aligned}A_n(\mathcal{T}\tilde{\mathbf{y}})_n &= \sum_{k\le n} \mathcal{A}(n+1,k)P_k(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &\quad - \sum_{k\ge n+1} \mathcal{A}(n+1,k)(\mathrm{Id} - P_k)(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}).\end{aligned}$$

Thus,

$$\begin{aligned}(\mathcal{T}\tilde{\mathbf{y}})_{n+1} - A_n(\mathcal{T}\tilde{\mathbf{y}})_n &= \sum_{k\le n+1} \mathcal{A}(n+1,k)P_k(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &\quad - \sum_{k\le n} \mathcal{A}(n+1,k)P_k(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &\quad - \sum_{k\ge n+2} \mathcal{A}(n+1,k)(\mathrm{Id} - P_k)(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &\quad + \sum_{k\ge n+1} \mathcal{A}(n+1,k)(\mathrm{Id} - P_k)(\tilde{y}_k - A_{k-1}\tilde{y}_{k-1}) \\ &= P_{n+1}(\tilde{y}_{n+1} - A_n\tilde{y}_n) + (\mathrm{Id} - P_{n+1})(\tilde{y}_{n+1} - A_n\tilde{y}_n) \\ &= \tilde{y}_{n+1} - A_n\tilde{y}_n.\end{aligned}$$

In particular, $\mathbf{w} = \mathcal{T}\tilde{\mathbf{y}} - \tilde{\mathbf{y}}$ is a solution of (3.1) satisfying $\|\mathbf{w}\| < +\infty$. Thus, it follows from our assumptions that $\mathbf{w} = \mathbf{0} = (0_n)_{n\in\mathbb{Z}}$. Consequently, $\mathcal{T}\tilde{\mathbf{y}} = \tilde{\mathbf{y}}$. Hence, since $\|\tilde{\mathbf{y}}\|_\infty \le L/(1-q) = C$, we get that $\tilde{\mathbf{y}} \in \mathcal{D}$ and, therefore, $\tilde{\mathbf{y}} = \tilde{\mathbf{x}} - \mathbf{z}$ is the unique fixed point of $\mathcal{T}$ in $\mathcal{D}$. The proof of the Theorem 3 is complete. □

Remark 3. Like in the continuous time setting, the uniqueness cannot hold, in general, if (3.1) has non-trivial bounded solutions and the reason is analogous: if $(x_n)_{n\in\mathbb{Z}}$ is a non-trivial bounded solution of (3.1), then $(0_n)_{n\in\mathbb{Z}}$ can be regarded as an approximate solution of (3.1) which can be shadowed by itself and any scalar multiple of $(x_n)_{n\in\mathbb{Z}}$.

3.3. **Examples.** As in the case of continuous time, Theorem 3 can be easily applied to the case when the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential dichotomy. In this particular case, the version of Theorem 3 follows from [3, Theorem 1].

However, in the following examples we will show that Theorem 3 is very flexible and that it can be applied to situations when the linear part does not exhibit good asymptotic behaviour.

Example 1. Let $(A_n)_{n\in\mathbb{Z}}$ be a sequence of isometries on a given Banach space $(X, |\cdot|)$ and let $(P_n)_{n\in\mathbb{Z}}$ be any sequence in $\mathcal{B}(X)$ satisfying $\|P_n\| \le D$ and $\|\mathrm{Id}-P_n\| \le D$ for every $n \in \mathbb{Z}$ and some constant $D > 0$. We note that we do not require that operators P_n are projections. In this case we have that $\|\mathcal{G}(m,n)\| \le D$ for every $m, n \in \mathbb{Z}$. Now, let $(c_n)_{n\in\mathbb{Z}}$ and $(\delta_n)_{n\in\mathbb{Z}}$ be any sequences of positive numbers such that

$$\sum_{n\in\mathbb{Z}} c_n < \frac{1}{D} \text{ and } \sum_{n\in\mathbb{Z}} \delta_n < +\infty.$$

Then, conditions (3.6) and (3.7) are satisfied and Theorem 3 may be applied.

Example 2. Let $(B_n)_{n\in\mathbb{Z}}$ be a sequence of isometries acting on a Banach space $(X, |\cdot|)$ and let $(\rho_n)_{n\in\mathbb{Z}}$ be any sequence of numbers satisfying $\rho_n \ge 1$ for every $n \in \mathbb{Z}$. We now consider sequences $(A_n)_{n\in\mathbb{Z}}$ and $(P_n)_{n\in\mathbb{Z}}$ acting on $X \times X$ given by

$$A_n = \begin{pmatrix} \frac{\rho_n}{\rho_{n+1}}\mathrm{Id} & 0 \\ 0 & B_n \end{pmatrix} \quad \text{and} \quad P_n = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix},$$

for $n \in \mathbb{Z}$. Then,

$$\mathcal{G}(m,n) = \begin{cases} \begin{pmatrix} \frac{\rho_n}{\rho_m}\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} & \text{for } m \ge n; \\ -\begin{pmatrix} 0 & 0 \\ 0 & \mathcal{B}(m,n) \end{pmatrix} & \text{for } m < n, \end{cases}$$

where

$$\mathcal{B}(m,n) = \begin{cases} B_{m-1}\cdots B_n & \text{for } m > n; \\ \mathrm{Id} & \text{for } m = n; \\ B_m^{-1}\dots B_{n-1}^{-1} & \text{for } m < n. \end{cases}$$

Consequently,

$$\|\mathcal{G}(m,n)\| = \begin{cases} \frac{\rho_n}{\rho_m} & \text{for } m \ge n; \\ 1 & \text{for } m < n. \end{cases}$$

Therefore, since $\rho_m \geq 1$ for every $m \in \mathbb{Z}$, if sequences $(c_n)_{n\in\mathbb{Z}}$ and $(\delta_n)_{n\in\mathbb{Z}}$ satisfy

$$\sum_{n=-\infty}^{+\infty} c_{n-1}\rho_n < 1 \text{ and } \sum_{n=-\infty}^{+\infty} \delta_{n-1}\rho_n < +\infty,$$

then conditions (3.6) and (3.7) hold and Theorem 3 can be applied. Finally, we observe that by taking an appropriate sequence $(\rho_n)_{n\in\mathbb{Z}}$ (for instance $\rho_n = 1 + |n|$ or $\rho_n = 1 + \log(1 + |n|)$), the difference equation $x_{n+1} = A_n x_n$, $n \in \mathbb{Z}$ does not admit an exponential dichotomy.

Remark 4. We now comment on the relationship between Theorem 3 and [4, Theorem 6.2]. On the one hand, the present result is stronger than the previous one in the sense that here the maps P_n do not need to be projections and, moreover, they do not need to commute with the dynamics. That is, they do not need to satisfy that $A_n P_n = P_{n+1} A_n$ for $n \in \mathbb{Z}$ (see Example 1). On the other hand, [4, Theorem 6.2] is more general than the present result in the sense that it allows for a third direction along which we do not have any control on the dynamics and, moreover, the operators A_n do not need to be invertible. Furthermore, [4, Theorem 6.2] holds for one-sided sequences while Theorem 3 holds for bilateral sequences. In particular, the results complement each other.

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