Model Updating in Conservative Second-Order Systems

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Organization of this talk

1. Introduction to the F.E.M.U problem
2. Orthogonality relations for generalized first-order pencils
3. Application to F.E.M.U
4. Numerical Example
5. Some insights for non-conservative systems
6. Conclusion

1. Introduction to the F.E.M.U problem

Definition: The Finite-element Model Updating problem is the problem of updating a finite-element generated second-order model (M, D, K) described by

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = Bu(t)$$
$$y(t) = C_1 q(t) + C_2 \dot{q}(t)$$

using modal data acquired from a physical vibration test so that innacurate modeling assumptions can be corrected in a new updated model $(\tilde{M}, \tilde{D}, \tilde{K})$.

Motivation:

Very often a few experimentally measured eigenvalues and eigenvectors of a vibrating structure do not match very well with those computed by Finite Element techniques.

Approach:

A vibration engineer then tries to update the theoretical Finite Element model, that is, the matrices (M, D, K).

Successful ??

YES: The updated theoretical model is valid for future use.
NO: Try a different update OR rebuild the whole model under more realistic assumptions.

2. Orthogonality relations for symmetric positive-definite generalized first-order pencils

An eigenvalue-eigenvector pair (λ, x) of a pair of matrices (A, B) (or equivalently, of a linear pencil $A - \lambda B$) verifies

$$(A - \lambda B)x = 0. \tag{1}$$

A very simple result in Linear Algebra is that if A and B are symmetric matrices and the matrices X and Λ are defined as above, then

$$X^T AX = D_1 \tag{2}$$

$$X^T BX = D_2. \tag{3}$$

Proof: Consider the eigendecomposition $AX = BX\Lambda$. Multiplying on the left by X^T gives

$$X^T AX = X^T BX\Lambda.$$

Since the left-hand side is symmetric, we have

$$(X^T BX)\Lambda = \Lambda(X^T BX) \tag{4}$$

and therefore $X^T BX$ commutes with a diagonal matrix having nonzero entries, and then $X^T BX$ is itself a diagonal matrix, say

$$X^T BX = D_2. \tag{5}$$

Therefore,

$$X^T AX = D_2\Lambda = D_1, \tag{6}$$

another diagonal matrix. □

Consequence in L.A. theory:
The Generalized Rayleigh Quotient theorem.

If $A, B \in \mathbb{R}^{n\times n}$ are symmetric matrices and $x \in \mathbb{R}^n$ is given, then

$$\lambda = r(x) = \frac{x^T A x}{x^T B x} \tag{7}$$

minimizes $f(x) = \|(A - \lambda B)x\|_B$, where $\|z\|_B = z^T B^{-1} z$.

□

In particular, if $x = x_i$, $i = 1, 2, \ldots, n$, are the eigenvectors of the pair (A, B) then

$$\lambda_i = \frac{x_i^T A x_i}{x_i^T B x_i}.$$

3. Applications to the Finite-element model updating problem

The F.E.M.U problem can be mathematically defined as follows:

Given a symmetric positive semidefinite model (M, D, K) with a set $\{\lambda_k, x_k\}$, $k = 1, \ldots, m$ of eigenvalues and corresponding eigenvectors, and a measured set $\{\sigma_k, y_k\}$, $k = 1, \ldots, m$ of natural frequencies and correspondent mode shapes, find an updated symmetric model $(\tilde{M}, \tilde{D}, \tilde{K})$ such that

- the subset $\{\lambda_k, x_k\}, k = 1, \ldots, m$ is replaced by $\{\sigma_k, y_k\}, k = 1, \ldots, m$ as eigenvalues and corresponding eigenvectors of the new model $(\tilde{M}, \tilde{D}, \tilde{K})$
- the remaining subset of $2n-m$ eigenvalues and corresponding eigenvectors of the new model $(\tilde{M}, \tilde{D}, \tilde{K})$ are the same as those of (M, D, K).

The methods presented in the literature, so far, cannot guarantee the second item above, that is, the invariance of the unmeasured spectrum.

It is said that they allow "spill-over" to happen.

In the following, we are gonna show how the results on orthogonality relations of a second-order pencil can be adapted to solve the *finite-element model updating* problem in the particular case that no damping forces are considered, that is, in the case $D = 0$.

Theorem 3.1 : *Orthogonality Relations for Symmetric Semidefinite Undamped Quadratic Pencil*

Let $P(\lambda) = \lambda^2 M + K$ be a symmetric semidefinite pencil with distinct nonzero eigenvalues and let (Λ, X) be a compact representation of the finite eigenstructure of this pencil, verifying

$$MX\Lambda^2 + KX = 0 \tag{8}$$

where $X \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathbb{C}^{n\times n}$ ($\Lambda^2 \in \mathbb{R}^{n\times n}$ is diagonal with nonpositive entries) under the convention that every pair of eigenvalues $\lambda = \pm i\ \alpha$ corresponding to an eigenvector x is represented only once when assembling in (8). This compact representation makes the matrices X and Λ have at most n rows and n columns, instead of their usual dimensions for the quadratic eigenvalue problem for the pencil $Q(\lambda) = \lambda^2 M + K$.

Then the matrices D_1 and D_2 defined by

$$D_1 = X^T M X \tag{9}$$

and

$$D_2 = X^T K X \tag{10}$$

are diagonal and

$$D_2 = -D_1 \Lambda^2. \tag{11}$$

□

Corollary 3.1: Suppose that the hypothesis of Theorem 5.1 holds, let the matrices X and Λ be partitioned as

$$X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \ , \ \Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix} \tag{12}$$

and assume that Λ_1 and Λ_2 do not have a common nonzero entry. Then

$$X_1^T M X_2 = 0 \tag{13}$$

and

$$X_1^T K X_2 = 0. \tag{14}$$

□

F.E.M.U via Direct Methods Using Modal Data

Assume that only m natural frequencies and corresponding mode shapes vectors are to be updated and let (Λ, X) be a finite compact representation of the eigenstructure of the model; therefore, matrices Λ and X satisfy (8) and the conventions established before hold. Partition Λ and X as follows:

$$\Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}, X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \tag{15}$$

where $\Lambda_1 \in \mathbb{C}^{m\times m}, X_1 \in \mathbb{C}^{n\times m}, \Lambda_2 \in \mathbb{C}^{(n-m)\times(n-m)}, X_2 \in \mathbb{C}^{n\times(n-m)}$, and such that

- (Λ_1, X_1) corresponds to the set of frequencies and mode shapes that needs to be updated
- (Λ_2, X_2) corresponds to the set of frequencies and mode shapes that is to remain unchanged.

Let $\Sigma_1^2 \in \mathbb{R}^{m\times m}$ denote the matrix that contains the information about the measured frequencies and let

$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix} \tag{16}$$

be the matrix of corresponding mode shapes, where $Y_{11} \in \mathbb{R}^{m\times m}$ and $Y_{12} \in \mathbb{R}^{(n-m)\times m}$. It is assumed that only Y_{11} is known; the method itself conveniently constructs Y_{12}.

Theorem 3.2 : Consider the positive semidefinite model (M, D, K) with no damping, that is, $D = 0$. Let matrices $\Lambda \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$, which represent the modal structure of the model, satisfy (8) and be partitioned as in (15). Suppose that the diagonal submatrices Λ_1 and Λ_2 do not have a common nonzero entry. Then, for every symmetric matrix $\Phi \in \mathbb{R}^{m\times m}$, the updated symmetric matrix $\tilde{K}$ defined by

$$\tilde{K} = K - MX_1\Phi X_1^T M \tag{17}$$

satisfies

$$MX_2\Lambda_2^2 + \tilde{K}X_2 = 0. \tag{18}$$

□

In other words, Theorem 3.2 states that the symmetric updating of K by (17) is guaranteed to produce no spill-over.

We now show how the symmetric matrix Φ can be chosen so that the measured eigenvalues and eigenvectors are contained in the updated model; that is, with such choice of Φ, the matrix $\tilde{K}$ is such that

$$MY_1\Sigma_1^2 + \tilde{K}Y_1 = 0. \tag{19}$$

Substituting the expressions of Y_1 from (16) and $\tilde{K}$ from (17) in (19), we have

$$M\begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}\Sigma_1^2 + K\begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix} = MX_1\Phi X_1^T M\begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}. \tag{20}$$

Assume that MX_1 has full rank. Then the QR factorization of this product defines orthogonal matrices $U_1 \in \mathbb{R}^{n\times m}$ and $U_2 \in \mathbb{R}^{n\times(n-m)}$ and an upper triangular matrix $Z \in \mathbb{R}^{m\times m}$ satisfying

$$MX_1 = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} Z \\ 0 \end{bmatrix}. \tag{21}$$

Let $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}$ and $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, where $M_1, K_1 \in \mathbb{R}^{n\times m}$ and $M_2, K_2 \in \mathbb{R}^{n\times(n-m)}$.

After premultiplication by $\begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$ and using (21) with the above partitioning of M and K, equation (20) can be rewritten as

$$\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}(M_1Y_{11} + M_2Y_{12})\Sigma_1^2 + \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}(K_1Y_{11} + K_2Y_{12}) = \begin{bmatrix} Z \\ 0 \end{bmatrix}\Phi X_1^T M \tag{22}$$

Therefore, a solution $\Phi \in \mathbb{R}^{m\times m}$ to (20) exists only if

$$U_2^T\left(M_1Y_{11}+M_2Y_{12}\right)\Sigma_1^2 = -U_2^T\left(K_1Y_{11}+K_2Y_{12}\right),$$

which is equivalent to

$$U_2^TM_2Y_{12}\Sigma_1^2 + U_2^TK_2Y_{12} = -U_2^T(K_1Y_{11}+M_1Y_{11}\Sigma_1^2). \quad (23)$$

Once this equation is solved for Y_{12}, we can form the matrix Y_1 using (16) and then compute $\Phi \in \mathbb{R}^{m\times m}$ from

$$Y_1^TMY_1\Sigma_1^2 + Y_1^TKY_1 = (Y_1^TMX_1)\Phi(Y_1^TMX_1)^T \qquad (24)$$

which was obtained by premultiplying (19) by Y_1^T. In principle, equation (24) gives just a least-squares solution of (23). However, once (23) is satisfied, (24) gives an exact solution of (19).

However, the symmetry of the solution Φ will only be guaranteed if the matrix Y_1 is updated before the computation of Φ:

$$Y_1^TMY_1 = D_1 \quad (25)$$
$$Y_1^TKY_1 = D_2 \quad (26)$$
$$(27)$$

where D_1 and D_2 are two diagonal matrices of order m.

Algorithm 3.2 *: Model Updating of an Undamped Symmetric Positive Semidefinite Model Using Incomplete Measured Data*

Input: The symmetric matrices $M, K \in \mathbb{R}^{n\times n}$; the set of m analytical frequencies and mode shapes to be updated; the complete set of m measured frequencies and mode shapes from the vibration test.

Output: Updated stiffness matrix $\tilde{K}$.

Assumption: $M = M^T \geq 0$ and $K = K^T \geq 0$.

Step 1: Form the matrices $\Sigma_1^2 \in \mathbb{R}^{m\times m}$ and $Y_{11} \in \mathbb{R}^{m\times m}$ from the available data. Form the corresponding matrices $\Lambda_1^2 \in \mathbb{R}^{m\times m}$ and $X_1 \in \mathbb{R}^{n\times m}$.

Step 2: Compute the matrices $U_1 \in \mathbb{R}^{n\times m}$, $U_2 \in \mathbb{R}^{n\times(n-m)}$, and $Z \in \mathbb{R}^{m\times m}$ from the QR factorization:

$$MX_1 = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} Z \\ 0 \end{bmatrix}.$$

Step 3: Partition $M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}, K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ where $M_1, K_1 \in \mathbb{R}^{n\times m}$.

Step 4: Solve the following matrix equation to obtain $Y_{12} \in \mathbb{R}^{(n-m)\times m}$:

$$U_2^T M_2 Y_{12} \Sigma_1^2 + U_2^T K_2 Y_{12} = -U_2^T [K_1 Y_{11} + M_1 Y_{11} \Sigma_1^2]$$

and form the matrix

$$Y_1 = \begin{bmatrix} Y_{11} \\ Y_{12} \end{bmatrix}.$$

Step 5: Compute the matrix $L \in \mathbb{R}^{m\times m}$ and the diagonal matrix $J \in \mathbb{R}^{m\times m}$ such that $LJL^T = Y_1^T M Y_1$ is a symmetric

(LDL^T) factorization of $Y_1^T M Y_1$. Update the matrix Y_1 by $Y_1 \leftarrow Y_1(L^{-1})^T$.

Step 6: Compute $\Phi \in \mathbb{R}^{m\times m}$ by solving the following system of equations:

$$(Y_1^T M X_1)\Phi(Y_1^T M X_1)^T = Y_1^T M Y_1(\Sigma_1)^2 + Y_1^T K Y_1 \ .$$

Step 7: Update

$$\tilde{K} = K - M X_1 \Phi X_1^T M.$$

4. Numerical Example: Consider the symmetric quadratic pencil $P(\lambda) = \lambda^2 M + K$ where

- Symmetric positive definite mass matrix $M \in \mathbb{R}^{66\times 66}$ is the *MatrixMarket* matrix *bcsstm02*.
- Symmetric dense stiffness matrix $K \in \mathbb{R}^{66\times 66}$ is the *MatrixMarket* matrix *bcsstk02*.

The smallest 5 eigenvalues pairs were computed using **Matlab**

$$\{ \pm\sqrt{43.2650}i \ \ , \pm\sqrt{43.8497}i \ \ , \pm\sqrt{49.4537}i \ \ , \pm\sqrt{565.6758}i \ \ , \pm\sqrt{570.6$$

and therefore

$$(\Lambda_1)^2 = \begin{bmatrix} -43.2650 & 0 & 0 & 0 & 0 \\ 0 & -43.8497 & 0 & 0 & 0 \\ 0 & 0 & -49.4537 & 0 & 0 \\ 0 & 0 & 0 & -565.6758 & 0 \\ 0 & 0 & 0 & 0 & -570.6518 \end{bmatrix}$$

and the corresponding 5 eigenvectors (matrix X_1) were fully computed (**Matlab** built-in function **eig**).

We choose

$$Y_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(\Sigma_1)^2 = \begin{bmatrix} -80 & 0 & 0 & 0 & 0 \\ 0 & -120 & 0 & 0 & 0 \\ 0 & 0 & -160 & 0 & 0 \\ 0 & 0 & 0 & -200 & 0 \\ 0 & 0 & 0 & 0 & -240 \end{bmatrix}.$$

The presented algorithm gives the following updating matrix Φ:

$$\Phi = \begin{bmatrix} -114.436272 & 8.842767 & -40.686540 & -29.990338 & -7.358168 \\ 8.842767 & -122.943140 & 10.005896 & 7.772494 & 64.330458 \\ -40.686540 & 10.005896 & -126.878651 & -13.794853 & -6.919918 \\ -29.990338 & 7.772494 & -13.794853 & 394.208825 & -5.427779 \\ -7.358168 & 64.330458 & -6.919918 & -5.427779 & 442.945273 \end{bmatrix}$$

The corresponding updated matrix $\tilde{K}$ can be shown to verify

$$\|MY_1(\Sigma_1)^2 + \tilde{K}Y_1\|_F = 1.4062 \times 10^{-9}$$
$$\|MX_2(\Lambda_2)^2 + \tilde{K}X_2\|_F = 1.8533 \times 10^{-10}$$

and therefore

- the eigenstructure (Σ_1, Y_1) was accurately assigned.
- The unmeasured eigenvalues and eigenvectors remained unchanged, that is, no spill over occurred.

The next figure shows the magnitudes of the differences between the entries of the original and updated stiffness matrices.

5. What about non-conservative S.O. systems ?

Datta, Elhay and Ram (1997): Orthogonality Relations for the Symmetric Quadratic Pencil

Let $P(\lambda) = \lambda^2 M + \lambda D + K$ be a symmetric pencil having distinct generalized eigenvalues and let (Λ, X) be a finite representation of the eigenstructure of this pencil. Let the matrices D_1 and D_2 be defined by

$$D_1 = (X\Lambda)^T M X\Lambda - X^T K X \tag{28}$$

and

$$D_2 = (X\Lambda)^T D X\Lambda + (X\Lambda)^T K X + X^T K X\Lambda \tag{29}$$

and

$$D_3 = (X\Lambda)^T M X + X^T M X\Lambda + X^T D X. \tag{30}$$

are diagonal matrices.

Consequence in theory: Rayleigh Quotient-like expressions

$$\lambda_i = \frac{x_i^T(\lambda_i^2 M - K)x_i}{x_i^T(2\lambda_i M + D)x_i} \tag{31}$$

$$-\lambda_i = \frac{x_i^T(\lambda_i^2 D + 2\lambda_i K)x_i}{x_i^T(\lambda_i^2 M - K)x_i} \tag{32}$$

$$-\lambda_i^2 = \frac{x_i^T(\lambda_i^2 D + 2\lambda_i K)x_i}{x_i^T(2\lambda_i M + D)x_i} \tag{33}$$

for $i = 1, 2, \ldots, n$.

Theorem 5.1:

Consider a positive semidefinite model (M, D, K) and let $\Lambda \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$ satisfy

$$MX\Lambda^2 + DX\Lambda + KX = 0 \tag{34}$$

and be partitioned as

$$\Lambda = \begin{bmatrix} \Lambda_1 & \\ & \Lambda_2 \end{bmatrix}, X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} \tag{35}$$

where $\Lambda_1 \in \mathbb{C}^{m\times m}, X_1 \in \mathbb{C}^{n\times m}, \Lambda_2 \in \mathbb{C}^{(n-m)\times(n-m)}, X_2 \in \mathbb{C}^{n\times(n-m)}$.

Suppose that Λ_1 and Λ_2 do not have a common eigenvalue. Let $\Phi \in \mathbb{C}^{m\times m}$ be any symmetric matrix. Define

$$\begin{aligned} \tilde{M} &= M - MX_1\Lambda_1\Phi(X_1\Lambda_1)^T M \\ \tilde{D} &= D + MX_1\Lambda_1\Phi X_1^T K + KX_1\Phi(X_1\Lambda_1)^T M \\ \tilde{K} &= K - KX_1\Phi X_1^T K. \end{aligned} \tag{36}$$

Then the updated model $(\tilde{M}, \tilde{D}, \tilde{K})$ is symmetric and the unmeasured frequencies and mode shapes of the original model remain unchanged; that is

$$\tilde{M}X_2\Lambda_2^2 + \tilde{D}X_2\Lambda_2 + \tilde{K}X_2 = 0\ .$$

6. Conclusion

- A method for Finite Element Model Updating of conservative second order systems is presented.
- This method is able to fit the vibration test data in the model the best way (least-squares sense) and is also guaranteed to preserve the frequencies and mode shapes that were not measured.
- The corresponding algorithm is rich in BLAS-3 computations, which brings high-performance strategies for its computational solution.
- Some insight for the solution in the damped case was also developed and further research is on the way.

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