BIJECTIVE PROOFS USING TWO-LINE MATRIX REPRESENTATIONS FOR PARTITIONS

Eduardo H. M. Brietzke
Instituto de Matemática – UFRGS
Caixa Postal 15080
91509–900 Porto Alegre, RS, Brazil
email: brietzke@mat.ufrgs.br

José Plínio O. Santos
IMECC-UNICAMP
C.P. 6065
13084-970 Campinas, SP, Brazil
email: josepli@ime.unicamp.br

Robson da Silva
ICE-UNIFEI
C.P. 50
37500-903 Itajubá-MG
email: rsilva@unifei.edu.br

Dedicated to George Andrews for his 70th birthday

Abstract

In this paper we present bijective proofs of several identities involving partitions by making use of a new way for representing partitions as two-line matrices. We also apply these ideas to give a combinatorial proof for an identity related to three-quadrant Ferrers graphs.

keywords: Partitions, Combinatorial identities,Lebesgue Identity, Ferrers Graph
MSC Primary–11P81, Secondary–05A19

1 Introduction

In this paper we present a number of results by making use of a new way of representing partitions as two-line matrices introduced in [8]. As one will see, one of the new notations for unrestricted partitions has more explicit information on the conjugate partition than what is given by the well-known Frobenius' symbol.

In Section 2, we present basically three distinct notations for unrestricted partitions that are given in [8]. In Section 3, we construct a bijection between the set of partitions into distinct odd parts with no parts equal to 1 and the set of self-conjugate three-quadrant Ferrers graphs. In the fourth section, we present combinatorial proofs for three identities and we establish bijections between several classes of partitions. One of the results resembles the Euler Pentagonal Number Theorem and another one is related to Ramanujan's partial theta function. In the final section, we provide a new bijective proof for the Lebesgue Identity based on the two-line matrix representation for partitions.

2 Representation of partitions by two-line matrices

A very well-known representation of a partition by a two-line matrix is by means of the Frobenius' symbol. The purpose of this section is to recall three different ways of representing partitions given in [8]. The main motivation for these representations comes from results obtained in [8].

Theorem A. (Theorem 8, [8]) *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_k \\ d_1 & d_2 & d_3 & \cdots & d_k \end{pmatrix}, \tag{2.1}$$

where

$$\begin{aligned} & c_k = 0\,, \quad d_k \neq 0, \\ & c_t = c_{t+1} + d_{t+1}, \quad \text{for any } t < k, \\ & n = \sum c_t + \sum d_t. \end{aligned} \tag{2.2}$$

A natural bijection between the two sets is given in [8]. Indeed, there are two different natural bijections between unrestricted partitions and two-line matrices satisfying (2.2). Perhaps the best way to describe them is by an example.

First bijection. The number k of columns of the matrix corresponds to the number of parts of the partition.

Consider, for example, the partition $\lambda = (6, 5, 2, 2)$ of 15. We associate to λ a 2×4 matrix A of the form given in Theorem A in such a way that the sum of the entries in each column of A are the parts of λ. We have no choice for the fourth column, but to pick $c_4 = 0$ and $d_4 = 2$. Since c_3 must be 2 and the entries of the third column must add up to 2, then $d_3 = 0$. By the same argument, we must have $c_2 = 2$ and $d_2 = 3$. Also, $c_1 = 5$ and $d_1 = 1$. The representation is

$$\lambda = 6 + 5 + 2 + 2 = \begin{pmatrix} 5 & 2 & 2 & 0 \\ 1 & 3 & 0 & 2 \end{pmatrix}.$$

To go from the matrix to the partition, we only have to add the entries in each column. As explained in [8], the second line of the matrix provides a complete description of the partition λ' conjugate to λ. In the above example, the second row $(1, 3, 0, 2)$ indicates that λ' contains one 1, three 2's, no 3's and two 4's.

Second bijection. The number k of columns of the matrix corresponds to the largest part of the partition.

To any positive integer j, we associate a $2 \times j$ matrix with the first $j-1$ entries equal to 1 in the first row, the last entry in the second row equal to 1, and the remaining entries equal to 0. For example, we have

$$1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad 3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \cdots$$

If $j \leq k$, given a $2 \times j$ matrix A and a $2 \times k$ matrix B, we perform the sum $A + B$ as if $k - j$ zero columns were added to the right of A. To the partition $\lambda = (6, 5, 2, 2)$ considered before, for instance, we associate the matrix

$$\begin{aligned} 6 + 5 + 2 + 2 &= \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Note that if we now add up the columns of the matrix obtained, we get the conjugate partition and, as before, the second line contains a description of the given partition, saying that there is no 1's, two 2's, no 3's, no 4's, one 5 and one 6.

We now present two new bijections between unrestricted partitions of n and certain classes of two-line matrices. It is shown in [8] that these sets have the same number of elements, but a natural bijection between them is not presented there.

In [8], for positive integers k and j, a set of nonnegative integers is defined by

$$A_{k,j} = \{ck + dj \mid c, d \geq 0\}.$$

Theorem B. (Theorem 11, [8]) *Let $f(n)$ be the number of partitions of n of the form $\lambda_1 + \lambda_2 + \cdots + \lambda_s$, with $\lambda_r \in A_{k,j}$, $c_s \neq 0$ and such that for $c_t k + d_t j$ and $c_{t+1}k + d_{t+1}j$ consecutive parts, $c_t \geq 2 + c_{t+1} + d_{t+1}$. Then,*

$$\sum_{n=0}^{\infty} f(n)q^n = \sum_{n=0}^{\infty} \frac{q^{kn^2}}{(q^j;q)_n (q^k;q)_n}\,. \tag{2.3}$$

In the particular case $k = j = 1$, denoting by $p(n)$ the number of partitions of n, since

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum_{n=0}^{\infty} p(n)q^n\,, \tag{2.4}$$

the following corollary is obtained.

Corollary C. (Corollary 12, [8]) *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form* (2.1) *where*

$$\begin{aligned} &c_k \neq 0\,, \\ &c_t \geq 2 + c_{t+1} + d_{t+1}, \quad \text{for any } t < k, \\ &n = \sum c_t + \sum d_t. \end{aligned} \tag{2.5}$$

We now construct a first new bijection for the set of unrestricted partitions, namely a bijection between the set of matrices of the form (2.1) satisfying (2.5) and the set of unrestricted partitions of n. This provides a direct proof of Corollary C.

Construction of a first bijection. We now construct a natural bijection between the set of unrestricted partitions of n and the set of matrices of the form (2.1) with non-negative integer entries satisfying (2.5).

It suffices, for each $k \geq 1$ fixed, to establish a bijection between the set of unrestricted partitions of n with Durfee square with side k and the set of matrices described above with the additional requirement that the matrix should have k columns.

For instance, a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ (as usual we suppose $\lambda_t \geq \lambda_{t+1}$) with Durfee square of side 3 is completely characterized once we declare how many times each one of the numbers 1, 2, and 3 appears as a part of λ, and if we also know the three largest parts λ_1, λ_2, and λ_3 of λ. We have to associate to λ a 2×3 matrix that gives a clue of these six numbers. Let

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \tag{2.6}$$

be such a matrix. By (2.5) we can write

$$c_3 = 1 + j_3\,, \tag{2.7}$$

$$c_2 = 3 + j_2 + j_3 + d_3 \tag{2.8}$$

$$c_1 = 5 + j_1 + j_2 + j_3 + d_2 + d_3. \tag{2.9}$$

with $j_1, j_2, j_3 \geq 0$. Hence, the matrix (2.6) may be rewritten as

$$\begin{pmatrix} 5{+}j_1{+}j_2{+}j_3{+}d_2{+}d_2 & 3{+}j_2{+}j_3{+}d_3 & 1{+}j_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \tag{2.10}$$

or, still, as the sum

$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ d_1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} d_2 & 0 & 0 \\ 0 & d_2 & 0 \end{pmatrix} + \begin{pmatrix} d_3 & d_3 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

$$+ \begin{pmatrix} j_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} j_2 & j_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} j_3 & j_3 & j_3 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.11}$$

From the above discussion, it follows that the partition λ can be characterized by the numbers

$d_1 \longrightarrow$ the number of times that 1 is a part of λ

$d_2 \longrightarrow$ the number of times that 2 is a part of λ

$d_3 \longrightarrow$ the number of times that 3 is a part of λ, not counting here eventual parts equal to 3 contained in the Durfee square

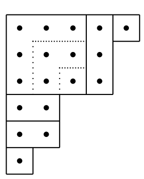
$j_1 \longrightarrow \lambda_1 - \lambda_2$, i.e., the number of times 1 is a part of the conjugate partition λ'

$j_2 \longrightarrow \lambda_2 - \lambda_3$, i.e., the number of times 2 is a part of λ'

$j_3 \longrightarrow \lambda_3 - 3$, i.e, the number of units that λ_3 exceeds the side of the Durfee square.

Note that the first three parts are completely determined: $\lambda_3 = 3 + j_3$, $\lambda_2 = 3 + j_2 + j_3$ and $\lambda_1 = 3 + j_1 + j_2 + j_3$.

Example 1. Consider the partition $\lambda = (5, 4, 4, 2, 2, 1)$ of $n = 18$.



We decompose the Ferrers' graph of λ as shown in the picture, taking into account the size of the Durfee square and the number of times 1, 2, and 3 are parts of λ and of the conjugate partition λ'. In the present example, $d_1 = 1$, $d_2 = 2$, $d_3 = 0$, $j_1 = 1$, $j_2 = 0$, and $j_3 = 1$. This decomposition suggests the sum of matrices below:

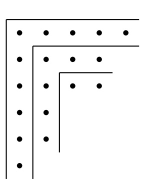
$\begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ from the Durfee square,

$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}$ because 1 appears once and 2 appears twice as parts of λ and

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ since 3 is twice part of λ', but appears only once outside of the Durfee square and 1 is part of λ' only once.

Hence, the matrix corresponding to the partition λ is

$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

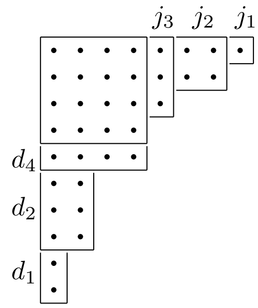


There is an easier way of obtaining the matrix-representation of the partition. We do not need to write out first the sum of matrices as done above. Indeed, in the example above, note that the sum of columns, 10, 6, and 2 are the length of the hooks shown in the picture to the left. In the second row, the entries 1 and 2 indicate the vertical displacement of the corresponding hook with respect to the previous one, and 0 indicates the vertical displacement of the last hook with respect to the bottom of the Durfee square.

The matrix $\begin{pmatrix} 5 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ corresponds to the Durfee square. Note that the entries 5, 3, and 1 of the first row are precisely the number of elements each hook has in common with the Durfee square.

Example 2. Determine the partition represented by the matrix $\begin{pmatrix} 15 & 9 & 5 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$.

The Durfee square has side four and the length of the hooks are $15 + 2 = 17$, $9 + 3 = 12$, $5 + 0 = 5$, and $1 + 1 = 2$. In addition, the second row of the matrix indicates that the first hook starts two units below the second hook, which in turn initiates three units below the third hook, the third hook begins at the same level than the fourth one, and this last hook begins one unit below the bottom of the Durfee square. This is all we need to know the Ferres' graph of the partition, which is shown in the picture to the right. The corresponding partition is $\lambda = (8, 7, 5, 4, 4, 2, 2, 2, 1, 1)$.



We now undertake the construction of a second bijection between unrestricted partitions and two-line matrices. Again, our motivation is a result presented in [8].

Theorem D. (Theorem 9, [8]) *The generating function for matrices of the form*

$$\begin{pmatrix} c_1 \times k & c_2 \times k & \cdots & c_s \times k \\ d_1 \times j & d_2 \times j & \cdots & d_s \times j \end{pmatrix},$$

where $d_t \neq 0$, $c_t \geq 1 + c_{t+1} + d_{t+1}$ and the sum of all entries is equal to n, is given by the LHS of the identity

$$\sum_{n=0}^{\infty} \frac{q^{kn^2+(j-k)n}}{(q^j;q^k)_n (q^k;q^k)_n} = \frac{1}{(q^j;q^k)_\infty}. \tag{2.12}$$

In the particular case $k = j = 1$, using (2.4), the following result follows.

Theorem E. (Theorem 10, [8]) *The number of unrestricted partitions of n is equal to the number of two-line matrices of the form*

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{2.13}$$

where

$$\begin{aligned} & d_t \neq 0, \\ & c_t \geq 1 + c_{t+1} + d_{t+1}, \\ & n = \sum c_t + \sum d_t\,. \end{aligned} \tag{2.14}$$

In [8], the equinumerability expressed in Theorem E is established by a generating function technique, but a natural bijection between the two sets is not presented.

Construction of a second bijection. We now construct a natural bijection between the set of unrestricted partitions of n and the set of matrices of the form (2.13) with non-negative integer entries satisfying (2.14). We rewrite the condition $c_t \geq 1 + c_{t+1} + d_{t+1}$ as

$$c_t = 1 + j_t + c_{t+1} + d_{t+1} \quad \forall t < k\,, \tag{2.15}$$

$$c_k = j_k\,, \tag{2.16}$$

$$j_t \geq 0. \tag{2.17}$$

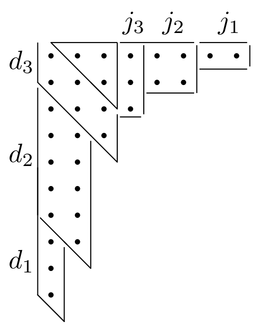
In this case we also take $s = k$ and define a bijection between the set of unrestricted partitions of n with Durfee square of side k and the set of matrices of the form (2.13) satisfying (2.15), (2.16) and (2.17), in addition to (2.14), with k columns.

For example, if $k = 3$, then

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{pmatrix} = \begin{pmatrix} 2+j_1+j_2+j_3+d_2+d_3 & 1+j_2+j_3+d_3 & j_3 \\ d_1 & d_2 & d_3 \end{pmatrix} \tag{2.18}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ d_1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} d_2 & 0 & 0 \\ 0 & d_2 & 0 \end{pmatrix} + \begin{pmatrix} d_3 & d_3 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

$$+ \begin{pmatrix} j_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} j_2 & j_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} j_3 & j_3 & j_3 \\ 0 & 0 & 0 \end{pmatrix}.$$



Compare (2.18) with the picture to the left. The entries d_1, d_2, and d_3 in the second row of the matrix are all non-vanishing and indicate the sizes of the subsets of the Ferrers' graph in the picture. The elements in the first row of the matrix, in turn, indicate the number of units still necessary to complete the hooks. The j_t's can vanish, but $d_t \neq 0$ for any t.

In the picture to the left, we show the example of the partition $\lambda = (8, 6, 4, 3, 2, 2, 2, 2, 1, 1)$, for which $d_1 = 3$, $d_2 = 5$, $d_3 = 2$, $j_1 = 2$, $j_2 = 2$ and $j_3 = 1$. The matrix associated to this partition is

$$\begin{pmatrix} 14 & 6 & 1 \\ 3 & 5 & 2 \end{pmatrix}.$$

The correspondence described by (2.18) or, alternatively, in a pictorial manner in the above picture, provides a new representation of unrestricted partitions by matrices. As with the representation obtained previously from (2.10), the number of columns in the matrix is equal to the side of the Durfee square of the partition.

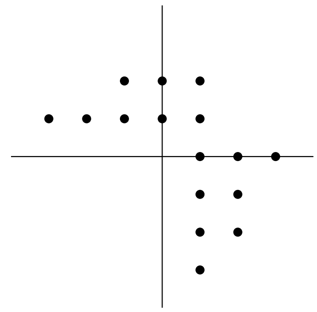
Note that by composition, we obtain a bijection between the set of matrices of the form (2.1) satisfying (2.5) and those satisfying (2.14). The bijection simply takes

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_k \\ d_1 & d_2 & \cdots & d_k \end{pmatrix} \longmapsto \begin{pmatrix} c_1 - 1 & c_2 - 1 & \cdots & c_k - 1 \\ d_1 + 1 & d_2 + 1 & \cdots & d_k + 1 \end{pmatrix}.$$

3 A combinatorial proof for an identity involving three-quadrant Ferrers graphs

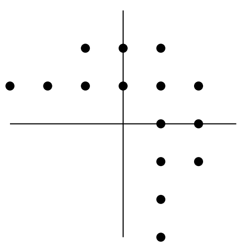
In [1], G. E. Andrews presented the three-quadrant Ferrers graphs as an extension of the two-quadrant Ferrers graphs. The three-quadrant Ferrers graphs of a positive integer n are constructed by placing n points in the first, second and fourth quadrants of the plane observing that each point must have at least one positive coordinate. We also require that the points on the x-axis and on the y-axis form the longest row and the tallest column among all rows and all columns in the set with positive x-coordinates and y-coordinates, respectively.

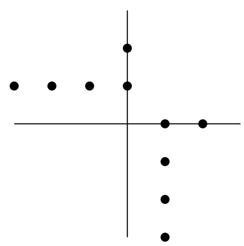
For example:



We recall from [1] that the self-conjugate three-quadrant Ferrers graphs are the ones that are unchanged by the mapping $(x, y) \mapsto (y, x)$. Let $s_3(n)$ denote the number of self-conjugate three-quadrant Ferrers graphs of n.

For example, the next two three-quadrant Ferrers graphs are self-conjugate:





In [1] the following identity is proved

$$\sum_{n=0}^{\infty} s_3(n) q^n = \prod_{n=0}^{\infty} \frac{(1+q^{2n+3})}{(1-q^{2n+2})}. \tag{3.1}$$

The RHS of (3.1) is the generating function for partitions in which the odd parts are distinct and greater than 1. We present in this section a bijective proof of (3.1) by using a two-line matrix representation for these partitions.

We now recall from [8] and [9] that we can represent a partition of n in which the odd parts are distinct and greater than 1 as a two-line matrix of the form

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{3.2}$$

where c_t and d_t satisfy $c_s = 2$ or 3, $d_t \equiv 0 \pmod 2$ and

- if $c_t \equiv 0 \pmod 2$ and $c_{t+1} \equiv 0 \pmod 2$, then $c_t = c_{t+1} + d_{t+1}$;
- if $c_t \equiv 0 \pmod 2$ and $c_{t+1} \equiv 1 \pmod 2$, then $c_t = 1 + c_{t+1} + d_{t+1}$;
- if $c_t \equiv 1 \pmod 2$ and $c_{t+1} \equiv 0 \pmod 2$, then $c_t = 1 + c_{t+1} + d_{t+1}$;
- if $c_t \equiv 1 \pmod 2$ and $c_{t+1} \equiv 1 \pmod 2$, then $c_t = 2 + c_{t+1} + d_{t+1}$,

with the sum of all entries equal to n. This same class of matrices is used in Section 4 with a different purpose.

We associate to each two-line matrix

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix},$$

a partition $\lambda_1 + \cdots + \lambda_s$ by just adding up the elements of the columns: $\lambda_t = c_t + d_t$. For example, the matrix $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 0 & 0 \end{pmatrix}$ is associated to $5 + 2 + 2$.

As the entries in the second line are even numbers, the parity of each part is determined by the corresponding entry of the first line. Then, it is impossible to have two equal consecutive odd parts $\lambda_t = c_t + d_t$ and $\lambda_{t+1} = c_{t+1} + d_{t+1}$, since we have $c_t = 2 + c_{t+1} + d_{t+1}$ in this case. As we begin with $c_s = 2$ or 3, each part, obtained by adding up the entries in corresponding column, is grater than 1.

We describe next how to go from a partition where the odd parts are distinct and greater than 1 to a two-line matrix satisfying the conditions above:

1. we start at the end of the first line, placing 2 or 3 if λ_s is even or odd, respectively. Then, we complete the column with an even number d_s such that the sum of the entries of this column is equal to λ_s. So the last column is uniquely determined.
2. in order to create the column just before the last one, we must observe the parity of λ_{s-1} and λ_s, because c_{s-1} and c_s have to satisfy the conditions above. For example, if c_{s-1} and c_s are both odd, then $c_{s-1} = 2 + c_s + d_s$ and we complete this column by choosing an even number d_{s-1} such that $\lambda_{s-1} = c_{s-1} + d_{s-1}$. It is not difficult to see that there is only one way to fill the column up.
3. to build the previous column we observe the parity of λ_{t-1} and λ_t and follow the procedure described in item 2.

For example, the table below presents the matrices and the partitions having odd parts distinct and greater than 1 for $n = 8$.

$\begin{pmatrix} 2 \\ 6 \end{pmatrix}$	8
$\begin{pmatrix} 2 & 2 \\ 4 & 0 \end{pmatrix}$	6+2
$\begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix}$	5+3
$\begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}$	4+4
$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \end{pmatrix}$	4+2+2
$\begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	2+2+2+2

Table 1: The matrices and partitions for $n = 8$

The bijective proof

It is convenient to rewrite the matrices of the form (3.2) as

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ 2d_1 & 2d_2 & \cdots & 2d_s \end{pmatrix}, \tag{3.3}$$

where c_t and d_t satisfy $c_s = 2$ or 3 and

- if $c_t \equiv 0 \pmod 2$ and $c_{t+1} \equiv 0 \pmod 2$, then $c_t = c_{t+1} + 2d_{t+1}$;
- if $c_t \equiv 0 \pmod 2$ and $c_{t+1} \equiv 1 \pmod 2$, then $c_t = 1 + c_{t+1} + 2d_{t+1}$;
- if $c_t \equiv 1 \pmod 2$ and $c_{t+1} \equiv 0 \pmod 2$, then $c_t = 1 + c_{t+1} + 2d_{t+1}$;
- if $c_t \equiv 1 \pmod 2$ and $c_{t+1} \equiv 1 \pmod 2$, then $c_t = 2 + c_{t+1} + 2d_{t+1}$.

Due to the restrictions on c_t, we have $c_t = i_t + c_{t+1} + 2d_{t+1}$, where $i_t \in \{0, 1, 2\}$, for $t = 1, 2, \ldots, s-1$ and $c_s = 2 + i_s$, where $i_s = 0, 1$. We can split (3.3) as the sum

$$\begin{pmatrix} 2+i_1+\cdots+i_s+2d_2+\cdots+2d_s & \cdots & 2+i_{s-1}+i_s+2d_s & 2+i_s \\ 2d_1 & \cdots & 2d_{s-1} & 2d_s \end{pmatrix}$$

$$= \begin{pmatrix} 2+i_1+\cdots+i_s & \cdots & 2+i_{s-1}+i_s & 2+i_s \\ 0 & \cdots & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} d_2+\cdots+d_s & \cdots & d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}$$

Note that the numbers $2 + i_1 + \cdots + i_s, \ldots, 2 + i_{s-1} + i_s, 2 + i_s$ form a non increasing sequence ending in either 2 or 3 and that there is no gap in the odd numbers in this sequence (if $2k+1$ is in the sequence, then so is every odd integer between 3 and $2k+1$).

For example, the matrix

$$\begin{pmatrix} 26 & 22 & 19 & 11 & 8 & 3 \\ 2 & 4 & 2 & 6 & 2 & 4 \end{pmatrix}$$

can be split as

$$\begin{pmatrix} 8 & 8 & 7 & 5 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 9 & 7 & 6 & 3 & 2 & 0 \\ 1 & 2 & 1 & 3 & 1 & 2 \end{pmatrix}. \tag{3.4}$$

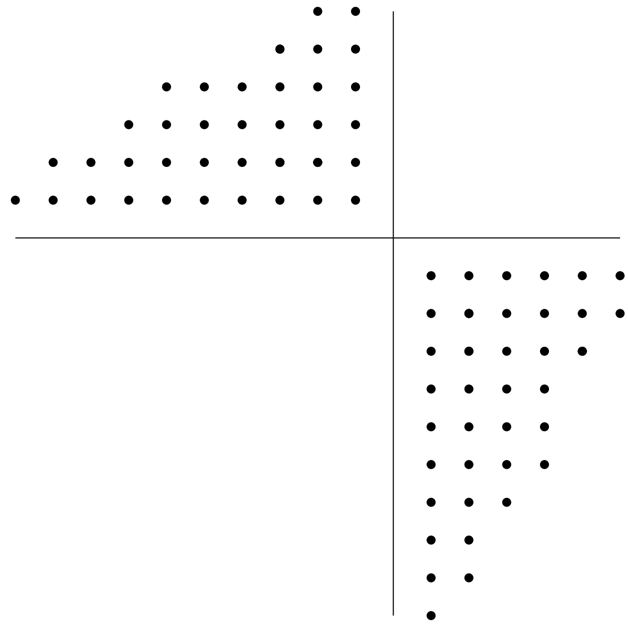
We will use this example to show how to associate a matrix like (3.3) and a three-quadrant Ferrers graph. The arguments in this example can be directly extended to the general case. We associate the matrix

$$\begin{pmatrix} 9 & 7 & 6 & 3 & 2 & 0 \\ 1 & 2 & 1 & 3 & 1 & 2 \end{pmatrix}$$

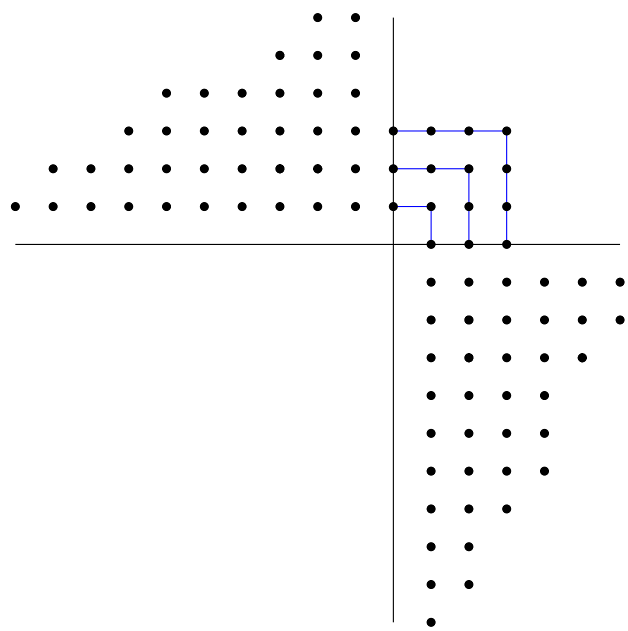
to the partition having one part equal to 1, two parts equal to 2, one part equal to 3, three parts equal to 4, one part equal to 5, and two parts equal to 6 $(6+6+5+4+4+4+3+2+2+1)$, whose Ferrers graph is

$$\begin{array}{llllll}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & & & & \\
\bullet & \bullet & & & & \\
\bullet & & & & &
\end{array}$$

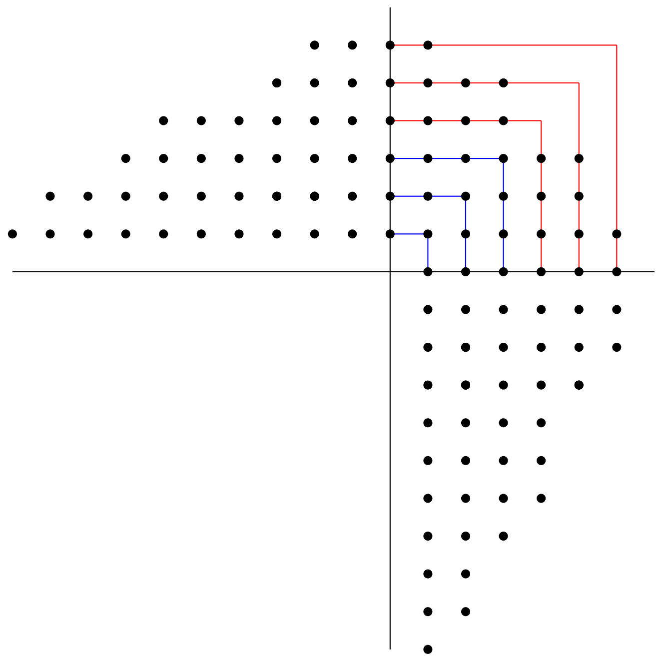
The pair of Ferrers graphs obtained from the second matrix in (3.4) can be placed in the plane in the following way



Now we describe how to place the contribution from the first matrix in (3.4) in the plane in order to obtain a self-conjugate three-quadrant Ferrers graph. To do this, we consider first the odd numbers in the first line: 7, 5, and 3. These numbers are represented in the first quadrant of the plane in the following way



The even numbers, $8, 8$ and 4, are represented in the first quadrant after the odd numbers in a non increasing sequence as follows



Note that the first even number after the greatest odd number, if there is one, is 1 plus the greatest odd number. Therefore, the construction of the three-quadrant Ferrers graph described above can always be done. It is clear that this procedure can be reversed in order to obtain the two-line matrix representing a partition in which the odd parts are disctinct and greater than 1.

The same argument applies in general and we have a bijection. Hence, identity (3.1) is proved.

4 Three identities and related bijections between certain classes of partitions

In this section we prove three partition identities ((4.1), (4.14), and (4.26) below) using combinatorial arguments and we establish bijections between several classes of partitions related to these identities. Our main tool is constructing bijections between certain classes of partitions and sets of two-line matrices. As an application of our results proved by combinatorial arguments, we obtain a straightforward proof of an identity about Ramanujan's ψ function (see (4.29) below), which is similar to dozens of identities for partial theta functions obtained by Ramanujan in his notebooks (see, for example, [3] and chapter 9 of [4]).

Our first objective is to study the identity

$$1+\sum_{k=1}^{\infty} \frac{(1+q)(1+q^3)\cdots(1+q^{2k-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, q^{2k} = \prod_{n=1}^{\infty} \frac{1+q^{2n+1}}{1-q^{2n}}\,. \tag{4.1}$$

We provide two combinatorial proofs for identity (4.1). As a matter of fact, we consider a slight improvement of (4.1).

Theorem 4.1 *The following identity holds*

$$1+\sum_{k=1}^{\infty} \frac{(1+zq)(1+zq^3)\cdots(1+zq^{2k-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, q^{2k} = \prod_{n=1}^{\infty} \frac{1+zq^{2n+1}}{1-q^{2n}}\,. \tag{4.2}$$

Proof: We introduce two classes of partitions

$\mathcal{P}_n$, the set of all partitions λ of n such that

– the odd parts of λ are distinct;

– the largest part of λ is even;

and

$\mathcal{Q}_n$, the set of all partitions μ of n such that

– the odd parts of μ are distinct;

– 1 is not a part of μ.

Identity (4.1) is equivalent to the statement that, for any n, $\mathcal{P}_n$ and $\mathcal{Q}_n$ have the same number of elements, or, in other words, there is a bijection between these two classes of partitions. Identity (4.2) suggests that it is possible to find such a bijection that preserves the number of odd parts of the partitions.

Denote by $\mathcal{P}_{n,k,m}$ the subset of all $\lambda \in \mathcal{P}_n$ such that λ contains exactly m odd parts and its largest part is $a(\lambda) = 2k$. Consider also the subset $\mathcal{Q}_{n,k,m}$ of all $\mu \in \mathcal{Q}_n$ such that μ has exactly m odd parts and its largest part is $a(\mu) = 2k$ or $2k+1$.

The coefficient of $z^m q^n$ on the left-hand-side and on the right-hand-side of (4.2) are, respectively, the number of elements of the sets

$$\mathcal{P}_{n,m} := \bigcup_k \mathcal{P}_{n,k,m} \qquad \text{and} \qquad \mathcal{Q}_{n,m} := \bigcup_k \mathcal{Q}_{n,k,m}. \tag{4.3}$$

Hence, to prove (4.2), it suffices to construct a bijection between the two classes of partitions $\mathcal{P}_{n,k,m}$ and $\mathcal{Q}_{n,k,m}$.

We now define a bijection $\varphi : \mathcal{P}_{n,k,m} \to \mathcal{Q}_{n,k,m}$. Given any partition $\lambda \in \mathcal{P}_{n,k,m}$, we define $\varphi(\lambda) \in \mathcal{Q}_{n,k,m}$ by:

– if 1 is not a part of λ, $\varphi(\lambda) = \lambda$;

– if 1 is a part of λ, $\varphi(\lambda)$ is the partition obtained by removing 1 from λ and adding one unit to its largest part $a(\lambda)$. If λ contains more than one copy of $a(\lambda)$, we add 1 to one of them only. Since $a(\lambda)$ is even, the number of odd parts is not altered.

Then, φ is a bijection with an inverse φ^{-1} defined by:

– if $\mu \in \mathcal{Q}_{n,k,m}$ is such that its largest part is even, then $\varphi^{-1}(\mu) = \mu$;

– if $\mu \in \mathcal{Q}_{n,k,m}$ and its largest part $a(\mu)$ is odd, $\varphi^{-1}(\mu)$ is obtained by removing from μ one part equal to $a(\mu)$ and adding one part equal to $a(\mu) - 1$ and one part equal to 1. $\square$

Of course, φ defined above can be extended to bijections from $\varphi : \mathcal{P}_{n,m} \to \mathcal{Q}_{n,m}$ or from $\varphi : \mathcal{P}_n \to \mathcal{Q}_n$.

Example. For $n = 11$, the two classes $\mathcal{P}_n = \mathcal{P}_{n,1}$ and $\mathcal{Q}_n = \mathcal{Q}_{n,1}$ defined in (4.3) and the correspondence φ constructed in Theorem 4.1 are given by

$$\begin{array}{ccccccc}
(10,1) & \longmapsto & (11) & \qquad & (6,2,2,1) & \longmapsto & (7,2,2) \\
(8,3) & \longmapsto & (8,3) & & (4,4,3) & \longmapsto & (4,4,3) \\
(8,2,1) & \longmapsto & (9,2) & & (4,4,2,1) & \longmapsto & (5,4,2) \\
(6,5) & \longmapsto & (6,5) & & (4,3,2,2) & \longmapsto & (4,3,2,2) \\
(6,4,1) & \longmapsto & (7,4) & & (4,2,2,2,1) & \longmapsto & (5,2,2,2) \\
(6,3,2) & \longmapsto & (6,3,2) & & (2,2,2,2,2,1) & \longmapsto & (3,2,2,2,2)
\end{array}$$

We now present another proof for identity (4.2) based on a two-line matrix representation for partitions. The idea of the proof is to introduce a certain class of two-line matrices and to interpret it in terms of partitions in two different ways. Consider the set $\mathcal{M}^{(2,3)}_{n,k,m}$ of $2 \times k$ matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_k \\ d_1 & d_2 & \cdots & d_k \end{pmatrix} \tag{4.4}$$

with non-negative integer entries, having exactly m odd entries in the first row, and satisfying

$$c_k \in \{2,3\}, \quad d_t \equiv 0 \ (\mathrm{mod}\ 2), \quad \forall t; \tag{4.5}$$

$$c_t \equiv 0 \ (\mathrm{mod}\ 2),\ c_{t+1} \equiv 0 \ (\mathrm{mod}\ 2) \Longrightarrow c_t = c_{t+1} + d_{t+1}; \tag{4.6}$$

$$c_t \equiv 0 \ (\mathrm{mod}\ 2),\ c_{t+1} \equiv 1 \ (\mathrm{mod}\ 2) \Longrightarrow c_t = 1 + c_{t+1} + d_{t+1}; \tag{4.7}$$

$$c_t \equiv 1 \ (\mathrm{mod}\ 2),\ c_{t+1} \equiv 0 \ (\mathrm{mod}\ 2) \Longrightarrow c_t = 1 + c_{t+1} + d_{t+1}; \tag{4.8}$$

$$c_t \equiv 1 \ (\mathrm{mod}\ 2),\ c_{t+1} \equiv 1 \ (\mathrm{mod}\ 2) \Longrightarrow c_t = 2 + c_{t+1} + d_{t+1}; \tag{4.9}$$

$$\sum c_t + \sum d_t = n\,. \tag{4.10}$$

This class of matrices was introduced in [9].

We now introduce a notation needed in the next theorem. Let $A \in \mathcal{M}^{(2,3)}_{n,k,m}$. If, for example, $k = 4$ and

$$A = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \in \mathcal{M}^{(2,3)}_{n,4,m}$$

then considering the restrictions (4.5)$-$(4.10) it is not difficult to verify that we can write in a unique manner $c_4 = 2 + j_4$, $c_3 = 2 + d_4 + j_3 + 2j_4$, $c_2 = 2 + d_3 + d_4 + j_2 + 2j_3 + 2j_4$, $c_1 = 2 + d_2 + d_3 + d_4 + j_1 + 2j_2 + 2j_3 + 2j_4$, with $j_t \in \{0,1\}$, i.e., A is of the form

$$A = \begin{pmatrix} 2+d_2+d_3+d_4+j_1+2j_2+2j_3+2j_4 & 2+d_3+d_4+j_2+2j_3+2j_4 & 2+d_4+j_3+2j_4 & 2+j_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

with $j_t \in \{0,1\}$. In the general case of a $2 \times k$ matrix $A \in \mathcal{M}^{(2,3)}_{n,k,m}$, we can express in a unique way

$$\begin{aligned} c_1 &= 2 + d_2 + d_3 + \cdots + d_k + j_1 + 2j_2 + \cdots + 2j_k \\ c_2 &= 2 + d_3 + d_4 + \cdots + d_k + j_2 + 2j_3 + \cdots + 2j_k \\ &\vdots \\ c_{k-1} &= 2 + d_k + j_{k-1} + 2j_k \\ c_k &= 2 + j_k \end{aligned} \tag{4.11}$$

where $j_t \in \{0,1\}$.

Theorem 4.2 *For any (n,k,m), with $0 \le m \le n$ and $k \ge 1$, there is a bijection*

$$\psi : \mathcal{M}^{(2,3)}_{n,k,m} \longrightarrow \mathcal{P}_{n,k,m}\,,$$

where $\mathcal{P}_{n,k,m}$ is the set of all partitions λ of n such that

– *λ has exactly m odd parts and the odd parts are distinct;*

– *the largest part of λ is $a(\lambda) = 2k$.*

Precisely, given $A \in \mathcal{M}^{(2,3)}_{n,k,m}$, $\lambda = \psi(A)$ is the partition containing:

– *$\frac{1}{2}\, d_t$ parts equal to $2t$, for any $t \in \{1,2,\ldots,k\}$;*

– *j_t parts equal to $2t-1$, for any $t \in \{1,2,\ldots,k\}$;*

– *one part equal to $2k$,*

where the $j_t \in \{0,1\}$ are defined by (4.11).

Proof: Let $j_t \in \{0,1\}$ as above. From (4.10) and (4.11) it follows that

$$n = 2k + (d_1 + 2d_2 + 3d_3 + \cdots + kd_k) + (j_1 + 3j_2 + 5j_3 + \cdots + (2k-1)j_k). \tag{4.12}$$

We define a partition $\lambda = \psi(A)$ containing:

– $\frac{1}{2}\, d_t$ parts equal to $2t$, for any $t \in \{1, 2, \ldots, k\}$;

– j_t parts equal to $2t-1$, for any $t \in \{1, 2, \ldots, k\}$;

– one part equal to $2k$.

Note that, for any $t \leq k$,

$$j_t = \begin{cases} 0, & \text{if } c_t \text{ is even} \\ 1, & \text{if } c_t \text{ is odd} \end{cases}$$

Therefore the number of odd parts in λ is equal to the number of odd elements in the first row of A.

It is easy to see that ψ defined above is a bijection between the set of matrices $\mathcal{M}^{(2,3)}_{n,k,m}$ and the set of partitions $\mathcal{P}_{n,k,m}$. $\square$

Denote by $\tilde{\mathcal{Q}}_{n,k,m}$ $(0 \leq m \leq n,\ k \geq 1)$ the set of partitions μ of n such that

– μ contains exactly k parts, m of these being odd;

– the odd parts of μ are distinct;

– 1 is not a part of μ.

Note that, for $\mathcal{Q}_{n,m}$ defined by (4.3), we have

$$\mathcal{Q}_{n,m} = \bigcup_k \mathcal{Q}_{n,k,m} = \bigcup_k \tilde{\mathcal{Q}}_{n,k,m}\,. \tag{4.13}$$

Our next result establishes a bijection between $\tilde{\mathcal{Q}}_{n,k,m}$ and the same class of matrices as before.

Theorem 4.3 *There is a natural bijection $\theta : \mathcal{M}^{(2,3)}_{n,k,m} \to \tilde{\mathcal{Q}}_{n,k,m}$. Given any matrix $A \in \mathcal{M}^{(2,3)}_{n,k,m}$ of the form* (4.4)*, the corresponding partition $\mu = \theta(A)$ is obtained adding up the entries in each column of A,*

$$\mu = (c_1 + d_1, c_2 + d_2, \ldots, c_k + d_k)\,.$$

Corollary 4.1 *Identity* (4.2) *holds true.*

Proof: As mentioned above, the cardinalities of the sets $\mathcal{P}_{n,m}$ and $\mathcal{Q}_{n,m}$, defined by (4.3), are, respectively, equal to the coefficient of $z^m q^n$ on left-hand-side and on the right-hand-side of (4.2). Since, by Theorem 4.2 and Theorem 4.3, $\mathcal{P}_{n,k,m}$ and $\tilde{\mathcal{Q}}_{n,k,m}$ are both in one-to-one correspondence with the same set of matrices $\mathcal{M}^{(2,3)}_{n,k,m}$, using (4.13) we obtain identity (4.2). $\square$

Application. We can construct a bijection $\theta \circ \psi^{-1} : \mathcal{P}_{n,k,m} \to \mathcal{Q}_{n,k,m}$, or $\theta \circ \psi^{-1} : \mathcal{P}_{n,m} \to \mathcal{Q}_{n,m}$, if we want, which is different from the bijection constructed in the proof of Theorem 4.1. For example, if $n = 11$ the bijection $\theta \circ \psi^{-1}$ is described below.

$$(2,2,2,2,2,1) \longmapsto \begin{pmatrix} 3 \\ 8 \end{pmatrix} \longmapsto (11)$$

$$(4,3,2,2) \longmapsto \begin{pmatrix} 4 & 3 \\ 4 & 0 \end{pmatrix} \longmapsto (8,3)$$

$$(4,4,3) \longmapsto \begin{pmatrix} 6 & 3 \\ 0 & 2 \end{pmatrix} \longmapsto (6,5)$$

$$(4,2,2,2,1) \longmapsto \begin{pmatrix} 3 & 2 \\ 6 & 0 \end{pmatrix} \longmapsto (9,2)$$

$$(4,4,2,1) \longmapsto \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \longmapsto (7,4)$$

$$(6,2,2,1) \longmapsto \begin{pmatrix} 3 & 2 & 2 \\ 4 & 0 & 0 \end{pmatrix} \longmapsto (7,2,2)$$

$$(6,4,1) \longmapsto \begin{pmatrix} 5 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix} \longmapsto (5,4,2)$$

$$(6,3,2) \longmapsto \begin{pmatrix} 4 & 3 & 2 \\ 2 & 0 & 0 \end{pmatrix} \longmapsto (6,3,2)$$

$$(6,5) \longmapsto \begin{pmatrix} 4 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto (4,4,3)$$

$$(8,2,1) \longmapsto \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix} \longmapsto (5,2,2,2)$$

$$(8,3) \longmapsto \begin{pmatrix} 4 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longmapsto (4,3,2,2)$$

$$(10,1) \longmapsto \begin{pmatrix} 3 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longmapsto (3,2,2,2,2)$$

We now study a second identity (see (4.14) below) that, surprisingly, has the same right-hand-side as (4.1).

Theorem 4.4 *The following identity holds*

$$\frac{1}{1-q^2} + \sum_{k=2}^{\infty} \frac{(1+q)(1+q^3)\cdots(1+q^{2k-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, q^{2k-1} = \prod_{n=1}^{\infty} \frac{1+q^{2n+1}}{1-q^{2n}}\,, \tag{4.14}$$

or, multiplying both sides by $1+q$*,*

$$\frac{1}{1-q} + \sum_{k=2}^{\infty} \frac{(1+q)^2(1+q^3)\cdots(1+q^{2k-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, q^{2k-1} = \prod_{n=1}^{\infty} \frac{1+q^{2n-1}}{1-q^{2n}}\,. \tag{4.15}$$

The proof of (4.14) depends on a relationship between two classes of partitions established in Theorem 4.5 below.

In addition to $\mathcal{Q}_n$, the set of partitions of n such that the odd parts are distinct and 1 is not a part, introduced in the proof of Theorem 4.1, we consider the following class of partitions:

$\mathcal{R}_n$, the set of all partitions λ of n such that

– the odd parts of λ are distinct;

– λ contains at least one odd part;

– if $a_o(\lambda)$ denotes the largest odd part, then $a_o(\lambda) + 1 \geq$ any even part.

Our next result resembles the Euler Pentagonal Number Theorem.

Theorem 4.5 *Let $\mathcal{R}_n$ and $\mathcal{Q}_n$ be the two classes of partitions defined above. Then,*

$$|\mathcal{Q}_n| - |\mathcal{R}_n| = (-1)^n \,. \tag{4.16}$$

Proof of Theorem 4.5: We construct a correspondence ϕ, which associates $\phi(\lambda) \in \mathcal{Q}_n$ to any given $\lambda \in \mathcal{R}_n$:

– if 1 is a part of λ, we define $\phi(\lambda)$ as the partition obtained by removing 1 from λ and adding 1 to its largest odd part;

– if 1 is not a part of λ, we take $\phi(\lambda) = \lambda$.

Then, ϕ is a one-to-one, with an inverse ϕ^{-1} defined by:

– if $\mu \in \mathcal{Q}_n$ is such that its largest odd part $a_o(\mu)$ and largest even part $a_e(\mu)$ satisfy $a_o(\mu)+1 \geq a_e(\mu)$, then $\phi^{-1}(\mu) = \mu$;

– if $\mu \in \mathcal{Q}_n$ contains no odd parts or if $a_o(\mu) + 1 < a_e(\mu)$, we construct $\phi^{-1}(\mu)$ by removing from μ one part equal to $a_e(\mu)$ and add one part equal to $a_e(\mu) - 1$ and one part equal to 1.

The correspondence ϕ defines a bijection except for the following two special features:

– when n is even, there is one element $\mu = (2, 2, \ldots, 2) \in \mathcal{Q}_n$ without a corresponding $\phi^{-1}(\lambda) \in \mathcal{R}_n$;

– when n is odd, there is one element $\lambda = (2, 2, \ldots, 2, 1) \in \mathcal{R}_n$ without a corresponding $\phi(\lambda) \in \mathcal{Q}_n$.

Therefore, (4.16) holds. $\square$

Proof of Theorem 4.4: It is easy to see that the generating function of $\mathcal{R}_n$ is

$$\frac{1}{1-q^2}\, q + \sum_{k=2}^{\infty} \frac{(1+q)(1+q^3)\cdots(1+q^{2k-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, q^{2k-1} \,. \tag{4.17}$$

From (4.16) and (4.17), we obtain

$$\frac{1}{1+q} + \frac{1}{1-q^2}\, q + \sum_{k=2}^{\infty} \frac{(1+q)(1+q^3)\cdots(1+q^{2k-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, q^{2k-1} = \prod_{n=1}^{\infty} \frac{1+q^{2n+1}}{1-q^{2n}} \,,$$

from which (4.14) follows immediately. $\square$

In order to construct a bijection between certain classes of partitions, we consider the set $\mathcal{M}^{(1)}_{n,k,m}$ of all $2 \times k$ matrices of the form (4.4), with non-negative integer entries, having exactly m odd entries in the first row, and satisfying

$$c_k = 1, \quad d_t \equiv 0 \pmod 2\,, \quad \forall t; \tag{4.18}$$

$$c_t \equiv 0 \pmod 2,\ c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = c_{t+1} + d_{t+1} \tag{4.19}$$

$$c_t \equiv 0 \pmod 2,\ c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 1 + c_{t+1} + d_{t+1} \tag{4.20}$$

$$c_t \equiv 1 \pmod 2,\ c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 1 + c_{t+1} + d_{t+1} \tag{4.21}$$

$$c_t \equiv 1 \pmod 2,\ c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 2 + c_{t+1} + d_{t+1} \tag{4.22}$$

$$\sum c_t + \sum d_t = n \,. \tag{4.23}$$

Let $A \in \mathcal{M}^{(1)}_{n,k,m}$. As before, if, for example, $k = 4$, then it is easy to verify that A is expressed in a unique way as

$$A = \begin{pmatrix} 2{+}d_2{+}d_3{+}d_4{+}j_1{+}2j_2{+}2j_3 & 2{+}d_3{+}d_4{+}j_2{+}2j_3 & 2+d_4+j_3 & 1 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix},$$

with $j_t \in \{0, 1\}$.

In general, for any $A \in \mathcal{M}^{(1)}_{n,k,m}$, we can write in a unique way

$$\begin{aligned} c_1 &= 2 + d_2 + d_3 + \cdots + d_k + j_1 + 2j_2 + \cdots + 2j_{k-1} \\ c_2 &= 2 + d_3 + d_4 + \cdots + d_k + j_2 + 2j_3 + \cdots + 2j_{k-1} \\ &\ \vdots \\ c_{k-1} &= 2 + d_k + j_{k-1} \\ c_k &= 1\,, \end{aligned} \tag{4.24}$$

where $j_t \in \{0, 1\}$.

Consider also the set $\mathcal{R}_{n,k,m}$ of partitions λ of n, having exactly m odd parts, such that the odd parts are distinct, the largest odd part is $a_o(\lambda) = 2k - 1$, and any even part is less than or equal to $2k$.

Theorem 4.6 *For any n, k, and m, with $1 \le m \le n$ and $k \ge 1$, there is a bijection $\zeta : \mathcal{M}^{(1)}_{n,k,m} \to \mathcal{R}_{n,k,m}$. Precisely, given $A \in \mathcal{M}^{(1)}_{n,k,m}$, $\lambda = \zeta(A)$ is the partition containing*

- *$\frac{1}{2}\, d_t$ parts equal to $2t$, for any $t \in \{1, 2, \ldots, k\}$;*
- *j_t parts equal to $2t - 1$, for any $t \in \{1, 2, \ldots, k - 1\}$;*
- *one part equal to $2k - 1$,*

where the $j_t \in \{0, 1\}$ are given by (4.24).

Remark. *In the case $k = 1$ (hence, n is necessarily odd) λ contains one part equal to 1 and $\frac{1}{2}d_1$ parts equal to 2.*

Proof: From (4.23) and (4.24) it follows that

$$n = (2k{-}1) + (d_1{+}2d_2{+}3d_3{+}\cdots{+}kd_k) + (j_1{+}3j_2{+}5j_3{+}\cdots{+}(2k-3)j_{k-1}). \tag{4.25}$$

To the matrix A we associate a partition λ containing:

- $\frac{1}{2}\, d_t$ parts equal to $2t$, for any $t \in \{1, 2, \ldots, k\}$;
- j_t parts equal to $2t - 1$, for any $t \in \{1, 2, \ldots, k - 1\}$;
- one part equal to $2k - 1$.

Note that, since $j_t = 1$ if c_t is odd and $j_t = 0$ if c_t is even, for any $t \le k - 1$, it follows that the number of odd parts in λ is equal to the number of odd elements in the first row of A. This completes the argument. □

Remark. In practice, to deal with the inverse correspondence ζ^{-1} it is useful to identify integers with matrices as described below. For simplicity, we consider the case $k = 4$.

$$1{=}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 3{=}\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 5{=}\begin{pmatrix} 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 7{=}\begin{pmatrix} 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$2{=}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, \quad 4{=}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \quad 6{=}\begin{pmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad 8{=}\begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

For example, the partition $\lambda = (8,7,4,4,4,3,1)$ corresponds to the matrix

$$\begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 13 & 5 & 4 & 1 \\ 0 & 6 & 0 & 2 \end{pmatrix}$$

In the case of the correspondence ψ considered in Theorem 4.2, the identification is the same, except for the integer $2k$. For example, for $k = 4$, remember that the largest part of any $\mu \in \mathcal{P}_{n,4,m}$ is $2k = 8$. The only difference is that the first 8 is identified with the matrix

$$\begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and each additional 8 is identified with

$$\begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

We now introduce another class of partitions. Let $\mathcal{S}_{n,k,m}$ denote the set of all partitions π of n such that

– π has exactly k parts and m of these parts are odd;

– the odd parts of π are distinct;

– the smallest part of π is odd.

Theorem 4.7 *There is a natural bijection $\xi : \mathcal{M}^{(1)}_{n,k,m} \to \mathcal{S}_{n,k,m}$. Given any matrix $A \in \mathcal{M}^{(1)}_{n,k,m}$, the corresponding partition $\pi = \xi(A)$ is defined as*

$$\pi = (c_1 + d_1, c_2 + d_2, \ldots, c_k + d_k)\,,$$

i.e., by adding up the entries in each column of A.

Proof: The proof is straightforward. □

Corollary 4.2 *There is a bijection $\xi \circ \zeta^{-1} : \mathcal{R}_{n,k,m} \to \mathcal{S}_{n,k,m}$.*

Proof: Combining the bijections given in in Theorem 4.6 and in Theorem 4.7, one obtains a bijection from $\xi \circ \zeta^{-1} : \mathcal{R}_{n,k,m} \to \mathcal{S}_{n,k,m}$. □

Example. Let $n = 11$. For $m = k = 1$, we have

$$(2,2,2,2,2,1) \quad \longmapsto \quad \begin{pmatrix} 1 \\ 10 \end{pmatrix} \quad \longmapsto \quad (11)$$

For $m = 1$ and $k = 2$, we have

$$(3,2,2,2,2) \quad \longmapsto \quad \begin{pmatrix} 2 & 1 \\ 8 & 0 \end{pmatrix} \quad \longmapsto \quad (10,1)$$

$$(4,3,2,2) \quad \longmapsto \quad \begin{pmatrix} 4 & 1 \\ 4 & 2 \end{pmatrix} \quad \longmapsto \quad (8,3)$$

$$(4,4,3) \quad \longmapsto \quad \begin{pmatrix} 6 & 1 \\ 0 & 4 \end{pmatrix} \quad \longmapsto \quad (6,5)$$

For $m = 1$ and $k = 3$, we have

$$
\begin{array}{cccccc}
(5,2,2,2) & \longmapsto & \left(\begin{array}{ccc} 2 & 2 & 1 \\ 6 & 0 & 0 \end{array}\right) & \longmapsto & (8,2,1) \\
(6,5) & \longmapsto & \left(\begin{array}{ccc} 4 & 4 & 1 \\ 0 & 0 & 2 \end{array}\right) & \longmapsto & (4,4,3) \\
(5,4,2) & \longmapsto & \left(\begin{array}{ccc} 4 & 2 & 1 \\ 2 & 2 & 0 \end{array}\right) & \longmapsto & (6,4,1) \\
(5,3,2,1) & \longmapsto & \left(\begin{array}{ccc} 5 & 3 & 1 \\ 2 & 0 & 0 \end{array}\right) & \longmapsto & (7,3,1)
\end{array}
$$

As an application of Theorem 4.7, we obtain the following identity.

Theorem 4.8 *The following identity holds*

$$\frac{q}{1-q^2} + \sum_{n=2}^{\infty} \frac{(1+q)(1+q^3)\cdots(1+q^{2n-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}\, q^{2n-1} = \sum_{n=1}^{\infty} q^{2n-1} \prod_{i=0}^{\infty} \frac{1+q^{2n+2i+1}}{1-q^{2n+2i}}\,. \tag{4.26}$$

Remark. Indeed a slightly stronger version of (4.26) holds true, namely

$$\frac{zq}{1-q^2} + \sum_{k=2}^{\infty} \frac{(1+zq)(1+zq^3)\cdots(1+zq^{2k-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, zq^{2k-1} = \sum_{k=1}^{\infty} zq^{2k-1} \prod_{i=0}^{\infty} \frac{1+zq^{2k+2i+1}}{1-q^{2k+2i}}\,. \tag{4.27}$$

Proof: The proof follows from Theorem 4.7 and Corollary 4.2. On one hand, the coefficient of $z^m q^n$ in the expansion of $\frac{(1+zq)(1+zq^3)\cdots(1+zq^{2k-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\, zq^{2k-1}$ is the number of elements of $\mathcal{R}_{n,k,m}$. Therefore, the coefficient of $z^m q^n$ in the left-hand-side of (4.27) is the number of elements in the union $\cup_k \mathcal{R}_{n,k,m}$.

On the other hand, if we denote by $\mathcal{S}_{n,m}$ the set of all partitions π of n such that

– the odd parts of π are distinct;

– π has exactly m odd parts;

– the smallest part of π is odd,

then the class $\mathcal{S}_{n,k,m}$, defined immediately before the statement of Theorem 4.7, consists of the partitions $\pi \in \mathcal{S}_{n,m}$ with exactly k parts. We consider also the class

$$\tilde{\mathcal{S}}_{n,k,m} = \{\pi \in \mathcal{S}_{n,m} |\text{ the smallest part of } \pi \text{ is } 2k-1\}.$$

Clearly,

$$\mathcal{S}_{n,m} = \bigcup_k \mathcal{S}_{n,k,m} = \bigcup_k \tilde{\mathcal{S}}_{n,k,m},$$

with the above unions being of pairwise disjoint sets.

The coefficient of $z^m q^n$ in the expansion of $zq^{2k-1} \prod_{i=0}^{\infty} \frac{1+zq^{2k+2i+1}}{1-q^{2k+2i}}$ is the number of elements of $\tilde{\mathcal{S}}_{n,k,m}$. Hence, the coefficient of $z^m q^n$ in the right-hand-side of (4.27) is the number of elements in the union $\mathcal{S}_{n,m} = \cup_k \mathcal{S}_{n,k,m} = \cup_k \tilde{\mathcal{S}}_{n,k,m}$. Since, by Corollary 4.2 of Theorem 4.7, $\mathcal{R}_{n,k,m}$ and $\mathcal{S}_{n,k,m}$ have the same number of elements, (4.27) follows. $\square$

It is important to point out that as a consequence of Theorem 4.8, that has been proved by a combinatorial argument, we can easily prove a well-known identity about Ramanujan's partial theta function

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}\,. \tag{4.28}$$

Theorem 4.9 *The function ψ, defined by (4.28), satisfies the identity*

$$1+q+\sum_{n=1}^{\infty}\frac{(1-q^2)(1-q^4)\cdots(1-q^{2n})}{(1-q^3)(1-q^5)\cdots(1-q^{2n+1})}\,q^{2n+1}=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\,. \tag{4.29}$$

Proof: From (4.26) it follows that

$$\frac{q}{1-q^2}+\sum_{n=2}^{\infty}\frac{(1+q)(1+q^3)\cdots(1+q^{2n-3})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}\,q^{2n-1}=\left(\prod_{i=1}^{\infty}\frac{1+q^{2i+1}}{1-q^{2i}}\right)\cdot U(q)\,,$$

where $U(q)$ is defined by

$$U(q):=q+\sum_{n=2}^{\infty}\frac{(1-q^2)(1-q^4)\cdots(1-q^{2n-2})}{(1+q^3)(1+q^5)\cdots(1+q^{2n-1})}\,q^{2n-1}\,.$$

Using (4.14), we obtain

$$\left(\prod_{i=1}^{\infty}\frac{1+q^{2i+1}}{1-q^{2i}}-\frac{1}{1+q}\right)=\left(\prod_{i=1}^{\infty}\frac{1+q^{2i+1}}{1-q^{2i}}\right)\cdot U(q)\,. \tag{4.30}$$

To prove (4.29) it suffices to verify that

$$1-U(-q)=\psi(q)=\sum_{n=0}^{\infty}q^{\frac{n(n+1)}{2}}\,. \tag{4.31}$$

Using the notation

$$(a;q)_{\infty}:=\prod_{i=0}^{\infty}(1-aq^i)\,,\qquad |q|<1\,,$$

we can rephrase (4.30) as

$$\frac{(-q^3;q^2)_{\infty}}{(q^2;q^2)_{\infty}}-\frac{1}{1+q}=\frac{(-q^3;q^2)_{\infty}}{(q^2;q^2)_{\infty}}\cdot\big(1-\psi(-q)\big)\,,$$

which is easily seen to be true, using (see (1.3.4), page 11, in [5])

$$\psi(q)=\frac{(q^2;q^2)_{\infty}}{(q;q^2)_{\infty}}\,.\qquad\square$$

Remark. Identity (4.2), and hence identity (4.1) as well, is a particular case of the following identity (see (1.5) in [2])

$$1+\sum_{k=0}^{\infty}\frac{(1+a)(1+aq)\cdots(1+aq^{k-1})}{(1-q)(1-q^2)\cdots(1-q^k)}\,z^kq^k=\prod_{j=1}^{\infty}\frac{1+azq^j}{1-zq^j}\,. \tag{4.32}$$

Indeed, taking $z=1$ in (4.32) and then replacing q by q^2 and a by z, one obtains (4.2) as a particular case.

In [2], (4.32) is proved by a combinatorial argument, but the proof is more involved than our proof of (4.2).

5 A new bijective proof for the Lebesgue Identity

We now turn our attention to the problem of obtaining a representation of the partitions in which even parts are not repeated as a certain class of two-line matrices. As a consequence of our ideas, we provide a new combinatorial proof of the following equality, known as the Lebesgue Identity:

$$\sum_{r=1}^{\infty}\frac{(1+zq)(1+zq^2)\cdots(1+zq^r)}{(1-q)(1-q^2)\cdots(1-q^r)}\,q^{\binom{r+1}{2}}=\prod_{i=1}^{\infty}(1+zq^{2i})(1+q^i). \tag{5.1}$$

Combinatorial proofs of the Lebesgue Identity have been given by several authors, among them Bessenrodt, Bressoud, Little, Sellers, Alladi and Gordon (see [6] and [7]). We believe our bijection is simpler. It is based on a new way of representing partitions by two-line matrices, introduced by Santos, Mondek, and Ribeiro in [8].

Let

$$f(z,q) := \sum_{r=1}^{\infty} \frac{(1+zq)(1+zq^2)\cdots(1+zq^r)}{(1-q)(1-q^2)\cdots(1-q^r)}\, q^{\binom{r+1}{2}} \tag{5.2}$$

be the left-hand side of (5.1). The general term

$$\frac{(1+zq)(1+zq^2)\cdots(1+zq^k)q^{1+2+3+\cdots+k}}{(1-q)(1-q^2)\cdots(1-q^k)}$$

of (5.2) is the generating function of the number of ways of decomposing n as

$$n = (1+j_1)\cdot 1 + (1+j_2)\cdot 2 + \cdots + (1+j_k)\cdot k + d_1 + 2d_2 + \cdots + kd_k\,,$$

with $d_t \geq 0$ and $j_t \in \{0,1\}$ or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} k+j_1+\cdots+j_k+d_2+\cdots+d_k & \cdots & 2+j_{k-1}+j_k+d_k & 1+j_k \\ d_1 & \cdots & d_{k-1} & d_k \end{pmatrix}.$$

The exponent of z counts the number of nonvaninhing elements among the j_t. Still equivalently, n is the sum of entries of the matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_k \\ d_1 & d_2 & \cdots & d_k \end{pmatrix}, \tag{5.3}$$

with non-negative entries and satisfying

$$c_k = i_k \in \{1,2\}\,, \tag{5.4}$$

$$c_t = i_t + c_{t+1} + d_{t+1}\,, \text{ with } i_t \in \{1,2\} \text{ for } t < k, \tag{5.5}$$

$$n = \sum c_t + \sum d_t\,. \tag{5.6}$$

Denote by $\mathcal{M}_{n,k}$ the set of all matrices of the form (5.3) with non-negative integer coefficients satisfying (5.4), (5.5), and (5.6). Denote by $\mathcal{M}_{n,m,k}$ the subset of all matrices in $\mathcal{M}_{n,k}$ for which $i_t = 2$ for exactly m elements in the first row and let $\mathcal{N}_{n,m} := \bigcup_k \mathcal{M}_{n,m,k}$. The following proposition clearly holds.

Proposition 5.1 *For each fixed k the coefficient of $z^m q^n$ in the expansion of*

$$\frac{(1+zq)(1+zq^2)\cdots(1+zq^k)}{(1-q)(1-q^2)\cdots(1-q^k)}\, q^{\binom{k+1}{2}}$$

is the number of matrices in the set $\mathcal{M}_{n,m,k}$. Hence, coefficient of $z^m q^n$ in the expansion of (5.2) *is the number of elements in $\mathcal{N}_{n,m}$.*

Denote by $\mathcal{P}_{n,m}$ the set of all partitions of n in which even parts are not repeated and having exactly m even parts. Our main result is the following theorem, from which the Lebesgue Identity will be derived.

Theorem 5.1 *There is a natural bijection between the sets $\mathcal{P}_{n,m}$ and $\mathcal{N}_{n,m}$.*

The construction of the bijection stated in Theorem 5.1 is the object of the next section.

Proposition 5.2 *The coefficient of $z^m q^n$ on the right-hand side of* (5.1) *is the number of elements of the set $\mathcal{P}_{n,m}$.*

Proof: It suffices to rewrite the right-hand side of (5.1) as

$$\prod_{i=1}^{\infty}(1+zq^{2i})(1+q^i) = \prod_{i=1}^{\infty}\frac{(1+zq^{2i})(1-q^{2i})}{(1-q^i)} = \prod_{i=1}^{\infty}\frac{(1+zq^{2i})}{(1-q^{2i-1})}. \qquad \square$$

We have interpreted the right-hand side and the left-hand side of the Lebesgue Identity (5.1) as generating functions of $\mathcal{N}_{n,m}$ and $\mathcal{P}_{n,m}$, respectively. Hence, to prove the Lebesgue Identity it suffices to construct a bijection $\psi : \mathcal{N}_{n,m} \longrightarrow \mathcal{P}_{n,m}$. This construction is the object of the next section.

6 Main bijection

In order to prove Theorem 5.1, we now construct a bijection $\psi : \mathcal{N}_{n,m} \longrightarrow \mathcal{P}_{n,m}$. As a matter of fact, we specialize and, for each k, construct a bijection $\psi : \mathcal{M}_{n,m,k} \longrightarrow \mathcal{P}_{n,m,k}$, where $\mathcal{M}_{n,m,k}$ is the set of matrices with k columns defined above and $\mathcal{P}_{n,m,k}$ is a subset of $\mathcal{P}_{n,m}$ to be defined below.

The definition of the set $\mathcal{P}_{n,m,k}$ and the bijection ψ is slightly different, according to the case that the number k of columns is even or odd.

Case 1: *k is even.* If $k = 2s$, we take $\mathcal{P}_{n,m,k}$ to be the subset of all partitions in $\mathcal{P}_{n,m}$ with exactly s parts greater than or equal to $k+1 = 2s+1$. Given any matrix $A \in \mathcal{M}_{n,m,k}$, using (5.4), (5.5), and (5.6) and setting $i_t = 1 + j_t$ $(t = 1, \dots, k)$, we have

$$\begin{aligned} n &= c_1 + \cdots + c_{2s} + d_1 + \cdots + d_{2s} \\ &= (j_1{+}2j_2{+}\cdots{+}2sj_{2s}) + (d_1{+}2d_2{+}\cdots{+}2sd_{2s}) + (1{+}2{+}\cdots{+}2s). \end{aligned}$$

Since $1 + 2 + \cdots + 2s = s(2s+1)$, we have

$$\begin{aligned} n =& \big(d_1{+}3d_3{+}\cdots{+}(2s{-}1)d_{2s-1}\big) + \big(2j_2{+}4j_4{+}\cdots{+}2sj_{2s}\big) + \big(j_1{+}2s{+}1{+}2d_{2s}\big) \\ &+ \big(3j_3{+}2s{+}1{+}2d_{2s-2}{+}2d_{2s}\big) + \cdots + \big((2s{-}1)j_{2s-1}{+}2s{+}1{+}2d_2{+}\cdots{+}2d_{2s}\big). \end{aligned}$$

Based on this decomposition, to the $2 \times (2s)$ matrix A we associate the partition π containing

(i) d_1 parts equal to 1;

(ii) d_3 parts equal to 3;

$\vdots$

(iii) d_{2s-1} parts equal to $2s-1$;

(iv) one part equal to $2r$, whenever $j_{2r} = 1$, $r \in \{1, 2, \dots, s\}$;

(v) one part equal to $j_1 + 2s + 1 + 2d_{2s}$;

(vi) one part equal to $3j_3 + 2s + 1 + 2d_{2s-2} + 2d_{2s}$;

$\vdots$

(vii) one part equal to $(2s-1)j_{2s-1} + 2s + 1 + 2d_2 + 2d_4 + \cdots + 2d_{2s}$.

It is easy to see that $\pi \in \mathcal{P}_{n,m,2s} = \mathcal{P}_{n,m,k}$.

The fact that ψ is a bijection from $\mathcal{M}_{n,m,k}$ onto $\mathcal{P}_{n,m,k}$ in Case 1, as well as in Case 2, follows from the following lemma.

Lemma 6.1 *Given k integers such that odd numbers are not repeated, there is a unique way to order* *them in a certain order $N_1, N_2, \ldots, N_k$ and write out*

$$\begin{aligned} N_1 &= 2e_1 + j_1 \\ N_2 &= 2e_2 + 3j_3 \\ N_3 &= 2e_3 + 5j_5 \\ &\cdots \\ N_k &= 2e_k + (2k-1)j_{2k-1} \end{aligned} \tag{6.1}$$

with

$$j_t \in \{0, 1\} \tag{6.2}$$

$$e_1 \le e_2 \le \cdots \le e_k \,. \tag{6.3}$$

To understand the idea of the proof of the above lemma, we provide an example.

Example. With $k = 8$, consider the following eight numbers, where odd numbers are not repeated: $(3, 4, 6, 6, 7, 11, 12, 17)$. First we express the even numbers as

$$\begin{aligned} 4 &= 2 \cdot 2 \\ 6 &= 2 \cdot 3 \\ 6 &= 2 \cdot 3 \\ 12 &= 2 \cdot 6 \end{aligned} \tag{6.4}$$

Then, we consider the least odd number in the set, which is 3. There is a unique way of expressing it, namely

$$3 = N_1 = 2 \cdot 1 + 1. \tag{6.5}$$

Adding (6.5) to the array (6.4), we obtain

$$\begin{aligned} 3 &= 2 \cdot 1 + 1 \\ 4 &= 2 \cdot 2 \\ 6 &= 2 \cdot 3 \\ 6 &= 2 \cdot 3 \\ 12 &= 2 \cdot 6 \end{aligned} \tag{6.6}$$

Then, we take the next odd number in the list, i.e., 7. There is only one way of fitting it into the array (6.6), namely at the second place, as N_2, with $j_3 = 1$,

$$\begin{aligned} 3 &= 2 \cdot 1 + 1 \\ 7 &= 2 \cdot 2 + 3 \\ 4 &= 2 \cdot 2 \\ 6 &= 2 \cdot 3 \\ 6 &= 2 \cdot 3 \\ 12 &= 2 \cdot 6 \end{aligned} \tag{6.7}$$

The next odd number in the list is 11 and the only way to fit it into the array (6.7) is between 4 and the first 6, i.e., as N_4, with $j_7 = 1$,

$$\begin{aligned} 3 &= 2 \cdot 1 + 1 \\ 7 &= 2 \cdot 2 + 3 \\ 4 &= 2 \cdot 2 \\ 11 &= 2 \cdot 2 + 7 \\ 6 &= 2 \cdot 3 \\ 6 &= 2 \cdot 3 \\ 12 &= 2 \cdot 6 \end{aligned} \tag{6.8}$$

Finally, the only way to fit the last odd number 17 into array (6.8) is in the sixth place, as N_6, with $j_{11} = 1$,

$$\begin{aligned}
3 &= 2\cdot 1 + 1\\
7 &= 2\cdot 2 + 3\\
4 &= 2\cdot 2\\
11 &= 2\cdot 2 + 7\\
6 &= 2\cdot 3\\
17 &= 2\cdot 3 + 11\\
6 &= 2\cdot 3\\
12 &= 2\cdot 6
\end{aligned} \tag{6.9}$$

The proof of the above lemma follows the same argument outlined in the example. First we arrange an array containing the expressions of the even elements in the list. Then, we begin including the odd elements one by one, in increasing order.

Case 2: *k is odd.* If $k = 2s+1$, we take $\mathcal{P}_{n,m,k}$ to be the subset of all partitions in $\mathcal{P}_{n,m}$ with at least $s+1$ parts greater than or equal to $k = 2s+1$, but at most s of them greater than or equal to $2s+3$. Given any matrix $A \in \mathcal{M}_{n,m,k}$, we have

$$\begin{aligned}
n &= c_1 + \cdots + c_{2s+1} + d_1 + \cdots + d_{2s+1}\\
&= \big(j_1{+}2j_2{+}\cdots{+}(2s{+}1)j_{2s+1}\big) + \big(d_1{+}2d_2{+}\cdots{+}(2s{+}1)d_{2s+1}\big) + \big(1{+}2{+}\cdots{+}(2s+1)\big).
\end{aligned}$$

Since $1 + 2 + \cdots + (2s+1) = (s+1)(2s+1)$, we have

$$\begin{aligned}
n =& \big(d_1{+}3d_3{+}\cdots{+}(2s{+}1)d_{2s+1}\big) + (2s{+}1{+}j_1) + \big(2j_2{+}4j_4{+}\cdots{+}2sj_{2s}\big) + \big(3j_3{+}2s{+}1{+}2d_{2s}\big)\\
&+ \big(5j_5{+}2s{+}1{+}2d_{2s-2}{+}2d_{2s}\big) + \cdots + \big((2s{+}1)j_{2s+1}{+}2s{+}1{+}2d_2{+}\cdots{+}2d_{2s}\big).
\end{aligned}$$

Based on this decomposition, to the $2\times(2s+1)$ matrix A we associate the partition π containing

- (i) d_1 parts equal to 1;
- (ii) d_3 parts equal to 3;

  $\vdots$
- (iii) d_{2s+1} parts equal to $2s+1$;
- (iv) One part equal to $2s+1+j_1$;
- (v) one part equal to $2r$, whenever $j_{2r} = 1$, $r \in \{1, 2, \ldots, k\}$;
- (vi) one part equal to $3j_3 + 2s + 1 + 2d_{2s}$;
- (vii) one part equal to $5j_5 + 2s + 1 + 2d_{2s-2} + 2d_{2s}$;

  $\vdots$
- $(viii)$ one part equal to $(2s+1)j_{2s+1} + 2s + 1 + 2d_2 + 2d_4 + \cdots + 2d_{2s}$.

It is easy to see that $\pi \in \mathcal{P}_{n,m,2s+1} = \mathcal{P}_{n,m,k}$.

We still have to show that any $\pi \in \mathcal{P}_{n,m}$ belongs to $\mathcal{P}_{n,m,k}$ for some k, i.e., $\mathcal{P}_{n,m} = \bigcup_k \mathcal{P}_{n,m,k}$. Indeed, given $\pi \in \mathcal{P}_{n,m}$, let s be the largest integer such that π has at least s parts greater than or equal to $2s+1$. If π has exactly s parts greater than or equal to $2s+1$, then $\pi \in \mathcal{P}_{n,m,2s}$, otherwise $\pi \in \mathcal{P}_{n,m,2s+1}$. This completes the proof ot Theorem 5.1. $\square$

The next example illustrates that our bijective proof of the Lebesgue Identity is different from the ones given in [6] and [7].

Example. Consider now the partition $\pi = (22, 21, 19, 18, 15, 10, 9, 9, 7, 4, 2)$, which is the same example as in [7], page 27. There are exactly five parts greater than or equal to 11. Hence, π corresponds to a

2×10 matrix. We first look at parts less than or equal to 10. There is one 7 and two 9's. Therefore $d_1 = d_3 = d_5 = 0$, $d_7 = 1$, and $d_9 = 2$. There is one of each, 2, 4, and 10. Hence, $j_2 = j_4 = j_{10} = 1$ and $j_6 = j_8 = 0$.

The five parts greater than or equal to 11 are 15, 18, 19, 21 and 22. For these five numbers, the representation given in Lemma 5.1 is

$$\begin{aligned} 15 &= 11 + 2 \cdot 2 \\ 18 &= 11 + 2 \cdot 2 + 3 \\ 22 &= 11 + 2 \cdot 3 + 5 \\ 19 &= 11 + 2 \cdot 4 \\ 21 &= 11 + 2 \cdot 5 \end{aligned}$$

and, hence,

$$\begin{aligned} d_{10} &= 2 \\ d_8 + d_{10} &= 2 \\ d_6 + d_8 + d_{10} &= 3 \\ d_4 + d_6 + d_8 + d_{10} &= 4 \\ d_2 + d_4 + d_6 + d_8 + d_{10} &= 5 \end{aligned}$$

It follows that $d_2 = 1$, $d_4 = 1$, $d_6 = 1$, $d_8 = 0$ and $d_{10} = 2$.

We now have all the elements to write out the matrix

$$A = \begin{pmatrix} 23 & 21 & 19 & 16 & 14 & 11 & 9 & 8 & 5 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 \end{pmatrix}.$$

Using the bijection constructed in Section 4, to this 2×10 matrix A we associate a pair (λ, μ) of partitions into distinct parts. We consider $j_t = c_t - c_{t+1} - d_{t+1} - 1$, if $t \leq k - 1$, and $j_k = c_k - 1$. In this concrete example, we have $(j_1, j_2, \ldots, j_{10}) = (0, 1, 1, 1, 1, 0, 0, 0, 0, 1)$. The first element of the pair is the partition λ formed with the non-vanishing elements in $(10j_{10}, 9j_9, \ldots, 2j_2, j_1)$, i.e., $\lambda = (10, 5, 4, 3, 2)$. Consider also the second row of A, $(d_1, d_2, \ldots, d_{10}) = (0, 1, 0, 1, 0, 1, 1, 0, 2, 2)$. The second element of the pair is the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{10})$ given by

$$\begin{aligned} \mu_{10} &= d_{10} + 1 \\ \mu_9 &= d_9 + d_{10} + 2 \\ &\vdots \\ \mu_1 &= d_1 + \cdots + d_{10} + 10 \end{aligned}$$

i.e., $\mu = (18, 17, 15, 14, 12, 11, 9, 7, 6, 3)$. This pair (λ, μ) is different from the one obtained in [7].

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