A new approach and generalizations to some results about Mock Theta Functions

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Abstract

Some Mock Theta functions have been interpreted in terms of n-color partition. In this paper we use a new technique to gain a deeper insight on these interpretations, as well as we employ this new technique to obtain in a more systematic way similar new interpretations for three other mock theta functions.

keywords: mock theta functions, n-color partitions, combinatorial interpretations, modular Ferrers diagrams

MSC Primary–11P81, Secondary–05A19

1 Introduction

In [1] combinatorial interpretations for the mock theta functions

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}, \tag{1.1}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n}, \tag{1.2}$$

$$\Phi_0(q) = \sum_{n=0}^{\infty} q^{n^2}(-q;q^2)_n, \tag{1.3}$$

and

$$\Phi_1(q) = \sum_{n=0}^{\infty} q^{(n+1)^2}(-q;q^2)_n \tag{1.4}$$

in terms of n-color partition are presented (see definitions and theorems stated below). In [2] a different combinatorial interpretation for these mock theta functions is given in terms of Bender-Knuth matrices.

Bender-Knuth matrices are infinite matrices with only finitely many non-vanishing entries, which have been introduced in [5], where a one-to-one correspondence with plane partitions is established.

In this paper we use a new technique to gain a deeper insight on these interpretations, as well as we employ this new technique to obtain in a more systematic way similar new interpretations for the mock theta functions

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(-q^2\,;q^4)_{n+1}}\,, \tag{1.5}$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}}\,, \tag{1.6}$$

and

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(q\,;q^2)_n}\,. \tag{1.7}$$

In [8], one of us, Santos, in a joint work with Mondek and Ribeiro, introduced a new combinatorial interpretation for partitions in terms of two-line matrices. In that paper a new way of representing, as two-line matrices, a number of identities from Slater's list ([9]) including Rogers-Ramanujan Identities, unrestricted partitions and Lebesgue's Partition Identity is described. In [6] we were able to provide a number of bijective proofs for several identities including a new bijective proof for the Lebesgue Identity. In that paper a combinatorial proof for an identity related to three-quadrant Ferrers graphs, given by Andrews (see [4]), was presented based on the two-line matrix representation. Even though our interpretation has not the same generality as that of Bender-Knuth, in [7] we have obtained interpretation for all the mock theta functions of [10] in terms of two-line matrices.

Our technique seems to be much more transparent than previous methods. As an application of our two-line matrix interpretation we were led in a natural way to new interpretations in terms of n-color partitions for the mock theta functions (1.5), (1.6), and (1.7).

Below we recall some definitions and state the interpretations obtained by Agarwal in [1] because they will be reobtained later as applications of our method. With this objective in mind, in Section 2 we obtain an equivalent characterization for n-color partitions. In Section 3, we introduce parameters and study a more general generating function including (1.1) and (1.2) as particular cases. We finish Section 3, by reobtaining Theorems 1.1 and 1.2 below. Our main goal in Section 4 is to provide new proofs for Theorems 1.3 and 1.4 based on representation by two-line matrices. In Section 5 we continue our investigation along the same lines and obtain a new combinatorial interpretation for the mock theta function (1.5) in terms of n-color partitions. In Section 6 we obtain new combinatorial interpretations for the mock theta functions (1.6) and (1.7) in terms of n-color partitions. Finally, in Section 7 we consider a family of generating functions containing several parameters and including as particular cases some of the generating functions studied before.

Definition 1.1 *An n-color partition (also called a partition with 'n copies of n') of a positive integer ν is a partition in which a part of size n can come in n different colors denoted by subscripts: $n_1, n_2, \ldots, n_n$ and the parts satisfy the order* $1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < \cdots$.

Example 1.1 *The n-color partitions of* 2 *and* 3 *are, respectively,*

$$2_1,\ 2_2,\ 1_1+1_1 \quad \textit{and} \quad 3_1,\ 3_2,\ 3_3,\ 2_1+1_1,\ 2_2+1_1,\ 1_1+1_1+1_1.$$

Definition 1.2 *The weighted difference of two parts $m_i, n_j, m \geq n$ is defined by $m-n-i-j$.*

Agarwal's interpretations of $\psi(q)$ and $F_0(q)$ are stated below.

Theorem 1.1 (see [1]) *For $\nu \geq 1$, let $A_1(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd parts with odd, for some k, k_k is a part, and the weighted difference of any two consecutive parts is* 0*. Then,*

$$\sum_{\nu=1}^{\infty} A_1(\nu)q^{\nu} = \psi(q). \tag{1.8}$$

Example 1.2 $A_1(8) = 3 : 8_8, 7_5 + 1_1, 6_2 + 2_2$.

Remark 1. It is important to mention that in the theorem above and in the next, the condition *"for some k, k_k is a part"* can be replaced by "the smallest part k has subscript k". It is equally important to mention that in Theorem 1.1 the statement that even parts appear with even subscripts and odd parts with odd is not really a conclusion of the theorem. It holds automatically for any n-color partition for which the smallest part k has subscript k and such that the weighted difference of consecutive parts is 0.

Theorem 1.2 (see [1]) *For $\nu \geq 1$, let $A_2(\nu)$ denote the number of n-color partitions of ν such that even parts appear with even subscripts and odd parts with odd greater than 1, for some k, k_k is a part, and the weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{\nu=1}^{\infty} A_2(\nu) q^{\nu} = F_0(q). \tag{1.9}$$

Theorem 1.3 (see [1]) *For $\nu \geq 1$, let $A_3(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, that is, the parts are of the type $(2k-1)_1$ or $(2k)_2$, the minimum part is 1_1 or 2_2, and the weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{\nu=0}^{\infty} A_3(\nu) q^{\nu} = \Phi_0(q). \tag{1.10}$$

Theorem 1.4 (see [1]) *For $\nu \geq 1$, let $A_4(\nu)$ denote the number of n-color partitions of ν such that only the first copy of the odd parts and the second copy of the even parts are used, the minimum part is 1_1, and the weighted difference of any two consecutive parts is 0. Then,*

$$\sum_{\nu=0}^{\infty} A_4(\nu) q^{\nu} = \Phi_1(q). \tag{1.11}$$

2 An equivalent characterization of n-color partitions

In this section we give two equivalent characterization for n-color partitions with no mention of subscripts. Throughout this section we consider n-color partitions $\lambda = (\lambda_1)_{\alpha_1} + (\lambda_2)_{\alpha_2} + \cdots + (\lambda_s)_{\alpha_s}$ such that

$$\lambda_s = \alpha_s\,; \tag{2.1}$$

$$\lambda_t - \lambda_{t+1} - \alpha_t - \alpha_{t+1} = 0\,, \ \forall t \ \text{(i.e., the weighted difference} \tag{2.2}$$

of consecutive parts is 0).

Remark 2. The subscripts α_t are completely determined by (2.1) and (2.2). Indeed,

$$\alpha_s = \lambda_s\,; \tag{2.3}$$

$$\alpha_t = \lambda_t - 2\lambda_{t+1} + 2\lambda_{t+2} - \cdots + (-1)^{s-t} 2\lambda_s\,, \quad \forall t < s. \tag{2.4}$$

Proposition 2.1 *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be a partition of n such that the sequence $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ defined by*

$$\begin{aligned} &\alpha_s = \lambda_s; \\ &\alpha_t = \lambda_t - 2\lambda_{t+1} + \cdots + (-1)^{s-t} 2\lambda_s, \quad \forall t < s. \end{aligned} \tag{2.5}$$

satisfies

$$\alpha_t \geq 1\,, \quad \forall t. \tag{2.6}$$

Then, $\lambda = (\lambda_1)_{\alpha_1} + \cdots + (\lambda_s)_{\alpha_s}$ *is an* n*-color partition of* n *satisfying the conditions (of Theorem 1.1)*

$$\alpha_s = \lambda_s; \tag{2.7}$$

$$1 \le \alpha_t \le \lambda_t\,, \quad \forall\, t; \tag{2.8}$$

$$\lambda_t \equiv \alpha_t \pmod 2, \quad \forall\, t; \tag{2.9}$$

$$\lambda_t - \lambda_{t+1} = \alpha_t + \alpha_{t+1}\,, \quad \forall\, t < s. \tag{2.10}$$

Conversely, any n*-color partition of* n *satisfying conditions* (2.7)–(2.10) *satisfies also equalities* (2.5) *above.*

The proof is simple, noting that solving (2.5) for λ_t yields

$$\begin{aligned} &\lambda_s = \alpha_s; \\ &\lambda_t = \alpha_t + 2\alpha_{t+1} + \cdots + 2\alpha_s, \quad \forall t < s. \end{aligned} \tag{2.11}$$

Remark 3. Condition (2.11) above shows that for the admissible partitions in Theorem 2 a more strict inequality holds, namely

$$1 \le \alpha_t \le \lambda_t - 2(s-t), \quad \forall\, t \le s. \tag{2.12}$$

We now give an interpretation in terms of modular Ferrers diagrams. It is easier to explain the idea by an example. Consider, for example, the partition $\lambda = (21, 14, 8, 3)$. By (2.5), we have $\alpha_1 = 3$, $\alpha_2 = 4$, $\alpha_3 = 2$, and $\alpha_4 = 3$ and, hence, λ satisfies the conditions (2.6). We represent the parts of λ as the sums of the elements of the columns of the following diagram

$$\begin{array}{lllll} 2 & 2 & 2 & 1 & \\ 2 & 2 & 2 & 1 & \alpha_4 = 3 \\ 2 & 2 & 2 & 1 & \\ 2 & 2 & 1 & & \\ 2 & 2 & 1 & \alpha_3 = 2 & \\ 2 & 1 & & & \\ 2 & 1 & & & \\ 2 & 1 & \alpha_2 = 4 & & \\ 2 & 1 & & & \\ 1 & & & & \\ 1 & \alpha_1 = 3 & & & \\ 1 & & & & \end{array}$$

It follows easily that the numbers α_t defined by (2.5) count the groups of 1s in the above diagram.

Therefore, partitions satisfying condition (2.6) are precisely those that can be represented by vertical lines in a diagram as above. We see immediately that the sums of the elements in the horizontal lines of the same diagram represent partitions of n into odd parts with no gaps. Therefore, we obtain the following theorem.

Theorem 2.1 *There is a bijection between the set of partitions of* n *satisfying* (2.6) *and the set of partitions of* n *into odd parts with no gaps. In other words, there is a bijection between the set of* n*-color partitions of* n *satisfying* (2.1) *and* (2.2) *and the set of partitions of* n *into odd parts with no gaps. Given an* n*-color partition* $\lambda = (\lambda_1)_{\alpha_1} + \cdots + (\lambda_s)_{\alpha_s}$ *of* n *satisfying* (2.1) *and* (2.2)*, the corresponding partition* μ *of* n *into odd parts with no gaps is the partition containing* α_t *parts equal to* $2t-1$, $\forall t \in \{1, \ldots, s\}$.

In addition, there is a bijection between each set of partitions above and the set of two-line matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{2.13}$$

with non-negative integer entries satisfying

$$c_s = 1\,; \tag{2.14}$$

$$d_t \ge 0\,, \quad \forall t; \tag{2.15}$$

$$c_t = 2 + c_{t+1} + 2d_{t+1}\,, \quad \forall t < s; \tag{2.16}$$

$$n = \sum c_t + \sum d_t. \tag{2.17}$$

Given a matrix A of the form (2.13)*, satisfying* (2.14)–(2.17) *above, the associated n-color partition λ satisfies $\lambda_t = c_t + d_t$ and $\alpha_t = 1 + d_t$, and the corresponding partition μ into odd parts with no gaps contains $1 + d_t$ parts equal to $2t - 1$, for any $t \leq s$.*

Furthermore, if λ is a partition of n and we define α_t by (2.5)*, then the weaker condition $\alpha_t \geq 0$, $\forall t$ is satisfied if and only if λ corresponds to a partition μ of n into odd parts in the way described above, possibly μ having some gaps.*

The connection between partitions and matrices stated in Theorem 2.1 is not proved here, since it will be generalized in Theorem 3.1 in Section 3.

As in Theorem 2.1, given a partition μ of n into odd parts, possibly having some gaps, we can still associate a partition λ to it, even though λ may not be an n-color partition. As an important particular case, we may consider the set of all partitions of n into odd parts and determine to which set of "generalized" n-color partitions of n it corresponds.

Let μ be any partition of n into odd parts. Let $2s-1$ be its largest part. Suppose μ contains d_t copies of $2t-1$, for any $t \leq s$ ($d_t \geq 0$, $d_s \geq 1$). Then,

$$n = 1d_1 + 3d_2 + 5d_3 + \cdots + (2s-1)d_s$$

or, equivalently, n can be expressed as the sum of the elements of the matrix

$$A = \begin{pmatrix} 2d_2 + \cdots + 2d_s & \cdots & 2d_{s-1} + 2d_s & 2d_s & 0 \\ d_1 & \cdots & d_{s-2} & d_{s-1} & d_s \end{pmatrix}. \tag{2.18}$$

We have the following result.

Proposition 2.2 *There is a bijection between the set of all partitions of n into odd parts and the set of two-line matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{2.19}$$

with non-negative integer entries satisfying

$$c_s = 0\,, \quad d_s \geq 1\,; \tag{2.20}$$

$$d_t \geq 0\,, \quad \forall t < s; \tag{2.21}$$

$$c_i = c_{i+1} + 2d_{i+1}\,, \quad \forall i < s; \tag{2.22}$$

$$n = \sum c_i + \sum d_i. \tag{2.23}$$

Given a matrix A the corresponding partition μ of n into odd parts is the partition μ containing d_t parts equal to $2t-1$, $\forall t \in \{1, \ldots, s\}$.

There is also a bijection between the set of partitions λ of n satisfying

$$\alpha_t \geq 0, \quad \forall t, \tag{2.24}$$

i.e.

$$\lambda_t - 2\lambda_{t+1} + 2\lambda_{t+2} - \cdots + (-1)^{s-t} 2\lambda_s \geq 0\,, \quad \forall t < s \tag{2.25}$$

and the same set of matrices of the form (2.19) *with non-negative integer entries satisfying* (2.20)–(2.23)*. Given a matrix A the corresponding partition λ of n is obtained adding up the elements of each of the columns of A.*

Example 3. For example, the partition $\mu = (11, 11, 9, 3, 3, 3, 1, 1, 1, 1)$ into odd parts corresponds to a matrix A of the form (2.19) in which the second row is $(4, 3, 0, 0, 1, 2)$. Hence, the matrix is

$$A = \begin{pmatrix} 12 & 6 & 6 & 6 & 4 & 0 \\ 4 & 3 & 0 & 0 & 1 & 2 \end{pmatrix}. \tag{2.26}$$

Adding up the elements in each column of (2.26), we obtain the partition $\lambda = (16, 9, 6, 6, 5, 2)$, which satisfies (2.24) or, equivalently, (2.25). The partition λ can also be obtained adding up the elements of each of the columns of the modular Ferrers diagram of μ:

$$
\begin{array}{llllll}
2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 1 & \\
2 & 1 & & & & \\
2 & 1 & & & & \\
2 & 1 & & & & \\
1 & & & & & \\
1 & & & & & \\
1 & & & & & \\
1 & & & & &
\end{array}
$$

Example 4. Below we construct a table showing the correspondence between partitions μ of n into odd parts and partitions λ of n satisfying (2.25) in the case $n = 10$. In this case there are 10 partitions of each type.

μ	matrix	λ	μ	matrix	λ
$(1,1,1,\ldots,1)$	$\begin{pmatrix} 0 \\ 10 \end{pmatrix}$	(10)	$(3,1,1,1,1,1,1,1)$	$\begin{pmatrix} 2 & 0 \\ 7 & 1 \end{pmatrix}$	$(9,1)$
$(3,3,1,1,1,1)$	$\begin{pmatrix} 4 & 0 \\ 4 & 2 \end{pmatrix}$	$(8,2)$	$(3,3,3,1)$	$\begin{pmatrix} 6 & 0 \\ 1 & 3 \end{pmatrix}$	$(7,3)$
$(5,1,1,1,1,1)$	$\begin{pmatrix} 2 & 2 & 0 \\ 5 & 0 & 1 \end{pmatrix}$	$(7,2,1)$	$(5,3,1,1)$	$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$	$(6,3,1)$
$(5,5)$	$\begin{pmatrix} 4 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$(4,4,2)$	$(7,1,1,1)$	$\begin{pmatrix} 2 & 2 & 2 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$	$(5,2,2,1)$
$(7,3)$	$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	$(4,3,2,1)$	$(9,1)$	$\begin{pmatrix} 2 & 2 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$(3,2,2,2,1)$

3 Generalization of Theorems 1 and 2

Theorem 3.1 *For fixed integers $i, k, r \geq 1$, consider the generating function*

$$\sum_{n=0}^{\infty} \frac{q^{rn^2}}{(q^i\,;q^k)_n}\,. \tag{3.1}$$

Then, the coefficient of q^n in the expansion of (3.1) *is the number of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{3.2}$$

with non-negative integer entries satisfying

$$c_s = r\,; \tag{3.3}$$

$$d_t \text{ are multiples of } i\,; \tag{3.4}$$

$$c_t = 2r + c_{t+1} + \frac{k}{i}\,d_{t+1}\,, \quad \forall t < s\,; \tag{3.5}$$

$$n = \sum c_t + \sum d_t\,. \tag{3.6}$$

Proof. For any s, the general term of (3.1),

$$\frac{q^{r(1+3+5+\cdots+(2s-1))}}{(1-q^i)(1-q^{i+k})\cdots(1-q^{i+(s-1)k})},$$

generates the partitions into parts of two colors[1], say blue and green, containing

(i) exactly r blue parts equal to each one of the odd numbers 1, 3, 5, ... , $2s-1$;

(ii) any number of green parts from i, $i+k$, $i+2k$, $\cdots$, $i+(s-1)k$.

Accordingly, we decompose n as

$$n = ie_1 + (i+k)e_2 + \cdots + (i+(s-1)k)e_s + r(1+3+5+\cdots+(2s-1)),$$

with $e_t \geq 0$, or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} (2s-1)r+ke_2+\cdots+ke_s & \cdots & 5r+ke_{s-1}+ke_s & 3r+ke_s & r \\ ie_1 & \cdots & ie_{s-2} & ie_{s-1} & ie_s \end{pmatrix}.$$

Then, the theorem follows, with $d_t = ie_t$. ■

We now specialize to the case in which $k = 2i$.

Corollary 3.1 *Given integers $r, i \geq 1$, the coefficient of q^n in the expansion of the generating function*

$$\sum_{n=0}^{\infty} \frac{q^{rn^2}}{(q^i\,;q^{2i})_n}. \tag{3.7}$$

is the number of matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{3.8}$$

with integer entries satisfying

$$c_s = r\,; \tag{3.9}$$

$$d_t \geq 0 \quad \textit{is a multiple of } i; \tag{3.10}$$

$$c_t = 2r + c_{t+1} + 2d_{t+1}, \quad \forall t < s\,; \tag{3.11}$$

$$n = \sum c_t + \sum d_t\,. \tag{3.12}$$

Remark 5. It follows from (3.9)–(3.11) that the matrix A is of the form

$$A = \begin{pmatrix} (2s-1)r+2ie_2+\cdots+2ie_s & \cdots & 5r+2ie_{s-1}+2ie_s & 3r+2ie_s & r \\ ie_1 & \cdots & ie_{s-2} & ie_{s-1} & ie_s \end{pmatrix}.$$

Proposition 3.1 *To any matrix of the form (3.8) satisfying (3.9)–(3.12) we associate a partition with subscripts $\lambda = (\lambda_1)_{\alpha_1} + (\lambda_2)_{\alpha_2} + \ldots + (\lambda_s)_{\alpha_s}$ by adding up the columns of the matrix, with*

$$\lambda_t = c_t + ie_t \qquad \textit{with a subscript} \qquad \alpha_t = r + ie_t\,.$$

Then, the partition λ has the following properties:

$$(i) \quad \alpha_t \equiv r \pmod{i}, \quad \forall t; \tag{3.13}$$

[1] strictly speaking it is not really necessary to consider two different colors, it just provides a nice setting for the argument

(*ii*) *The smallest part* λ_s *coincides with its subscript*

$$\alpha_s = \lambda_s; \tag{3.14}$$

(*iii*) *The weighted difference of any two consecutive parts is 0,*

$$\lambda_t - \lambda_{t+1} - \alpha_t - \alpha_{t+1} = 0. \tag{3.15}$$

$$(iv) \quad r \le \alpha_t \le \lambda_t \quad \textit{and} \quad \lambda_t \equiv 2r(s-t) + \alpha_t \pmod{i}. \tag{3.16}$$

The proof will not be given here and follows from the same type of argument as Proposition 2.2.

We now state without proof a result giving a different characterization of partitions satisfying (3.13)–(3.16).

Proposition 3.2 *Let* $\lambda = (\lambda_1)_{\alpha_1} + (\lambda_2)_{\alpha_2} + \cdots + (\lambda_s)_{\alpha_s}$ *be a partition of* n *with subscripts satisfying* (3.13)–(3.16)*. Then,*

$$\begin{aligned} &0 \le \lambda_s - r \equiv 0 \pmod{i}; \\ &0 \le \lambda_t - 2\lambda_{t+1} + \cdots + 2(-1)^{s-t}\lambda_s - r \equiv 0 \pmod{i}, \quad \forall t < s. \end{aligned} \tag{3.17}$$

Application 1. There is a bijection between the following sets of partitions of n:

(*i*) Partitions with subscripts $\lambda = (\lambda_1)_{\alpha_1} + (\lambda_2)_{\alpha_2} + \cdots + (\lambda_s)_{\alpha_s}$ satisfying (3.13)–(3.16);

(*ii*) Partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ satisfying (3.17);

(*iii*) Partitions into parts of two colors, say blue and green, in which

(*a*) the blue parts are all odd, with no gaps, and there are exactly r copies of each one of them;

(*b*) the green parts are of the form $i\delta$, with δ an odd number less than or equal to the largest blue part.

Also there is a bijection between any one of these classes of partitions of n and the set of matrices of the form (3.8) satisfying (3.9)–(3.12).

The generating function for any one of the above classes of partitions or matrices is (3.7).

In the particular case $i = 1$, the congruences mod i become trivial and we have that (3.7) is the generating function for the number of partitions of n such that

(*i*) The smallest part λ_s coincides with its subscript α_s and is equal to r;

(*ii*) For any other subscript α_t, we have $r \le \alpha_t \le \lambda_t$;

(*iii*) The weighted difference between any two consecutive parts is 0, i.e., $\lambda_t - \lambda_{t+1} - \alpha_t - \alpha_{t+1} = 0$, $\forall t$.

These results in the particular case $r = 1$ and $r = 2$ have been obtained by Agarwal in [1] and are the results referred to in the Introduction as Theorems 1.1 and 1.2. We rephrase this in the next theorem.

Theorem 3.2 (Generalization of Theorems 1 and 2) *If* $r \ge 1$ *is an integer, then the coefficient of* q^n *in the expansion of the generating function*

$$\sum_{n=0}^{\infty} \frac{q^{rn^2}}{(q\,;q^2)_n}\,. \tag{3.18}$$

is the number of elements in the set $\mathcal{M}(n,r)$ *of all matrices of the form* (3.8) *with non-negative integer entries satisfying*

$$c_s = r\,; \tag{3.19}$$

$$d_t \ge 0\,; \tag{3.20}$$

$$c_t = 2r + c_{t+1} + 2d_{t+1}\,, \quad \forall t < s. \tag{3.21}$$

There is a bijection between the set $\mathcal{M}(n,r)$ and the set of the partitions $\lambda = (\lambda_1, \dots, \lambda_s)$ of n satisfying

$$\begin{aligned} \lambda_s &\geq r; \\ \lambda_t - 2\lambda_2 + \cdots + 2(-1)^{s-t}\lambda_s &\geq r, \quad \forall t < s. \end{aligned} \tag{3.22}$$

Given any matrix $A \in \mathcal{M}(n,r)$, the corresponding partition λ is obtained by adding up the elements of each column of A.

Finally, there is also a bijection between the set $\mathcal{M}(n,r)$ and the set of the partitions μ of n into odd parts, with no gaps, such that each part has multiplicity at least r and the largest part is $2s-1$. Given any matrix $A \in \mathcal{M}(n,r)$, the corresponding μ is the partition containing $r + d_t$ copies of $2t-1$, for any $t \in \{1, 2, \dots, s\}$.

4 Revisiting Theorems 3 and 4 of Agarwal

At the end of Section 3, we proved Theorem 3.2, containing as particular cases Theorem 1.1 and Theorem 1.2 of Agarwal, stated in the Introduction. In this section we apply the method of two-line matrices to obtain new proofs for Theorem 1.3 and Theorem 1.4.

Consider the mock theta function $\Phi_0(q)$ of order 5 already defined in (1.3)

$$\Phi_0(q) = 1 + \sum_{s=1}^{\infty} (1+q)(1+q^3)\cdots(1+q^{2s-1})\, q^{1+3+5+\cdots+(2s-1)}\,.$$

It is clearly the generating function for the partitions of n into odd parts with no gaps and such that any part appears at most twice. Hence, using Theorem 2.1, Theorem 1.3 follows.

To get a better understanding, write out

$$n = (1+j_1)\cdot 1 + (1+j_2)\cdot 3 + (1+j_3)\cdot 5 + \cdots + (1+j_s)\cdot(2s-1)\,,$$

with $j_t \in \{0, 1\}$, or, in matrix form, n is the sum of the entries of the matrix

$$A = \begin{pmatrix} (2s-1) + 2j_2 + \cdots + 2j_s & \cdots & 3 + 2j_s & 1 \\ j_1 & \cdots & j_{s-1} & j_s \end{pmatrix}. \tag{4.1}$$

Therefore (1.3) is the generating function for the set of two-line matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{4.2}$$

with non-negative integer entries satisfying

$$c_s = 1\,; \tag{4.3}$$

$$d_t \in \{0, 1\}\,; \tag{4.4}$$

$$c_t = 2 + c_{t+1} + 2d_{t+1}\,, \quad \forall t < s\,; \tag{4.5}$$

$$\sum c_t + \sum d_t = n\,. \tag{4.6}$$

There is a bijection between the set of two-line matrices satisfying (4.3)–(4.6) and the set of partitions into odd parts, with no gaps, containing at most two copies of any part. The matrix A has s columns if and only if the largest part of the corresponding partition is $2s-1$. Furthermore, for any entry d_t in the second line we have that $d_t = 0$ or $d_t = 1$ according to the fact that $2t-1$ appears once or twice as a part of the partition.

Adding up the elements in each column of (4.1), we obtain a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ of n satisfying:

$$\lambda_s \in \{1, 2\} \tag{4.7}$$

$$\lambda_t - \lambda_{t+1} = \begin{cases} 2, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ are both odd;} \\ 3, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ have different parity;} \\ 4, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ are both even.} \end{cases} \tag{4.8}$$

We obtain the following theorem.

Theorem 4.1 *The mock theta function of order 5* (1.3)

$$\Phi_0(q) = \sum_{n=0}^{\infty} (-q\,; q^2)_n q^{n^2}$$

is the generating function for the number of two-line matrices with non-negative integer entries satisfying (4.3)–(4.6)*. Furthermore, there is a bijection between this set of matrices and each one of the sets of*

(*i*) *partitions λ of n satisfying* (4.7) *and* (4.8)*;*

(*ii*) *partitions μ of n into odd parts, with no gaps, containing at most two copies of any part.*

(*iii*) *n-color partitions $\lambda = (\lambda_1)_{\alpha_1} + \ldots + (\lambda_s)_{\alpha_s}$ satisfying:*

- (*a*) *the weighted difference of consecutive parts is 0, i.e., $\lambda_t - \lambda_{t+1} - \alpha_t - \alpha_{t+1} = 0$;*
- (*b*) *the smallest part λ_s is 1 or 2;*
- (*c*) *$\alpha_t = 1$ if λ_t is odd and $\alpha_t = 0$ if λ_t is even.*

Given any matrix A, the corresponding partition λ is obtained by adding up the entries in each of the columns of A and μ is the partition into odd parts containing one (respectively, two) copies of a number $2t - 1$ if $d_t = 0$ (respectively, $d_t = 1$) for any $t \le s =$ number of columns of A.

Example 5. For example, $n = 17$ has two partitions into odd parts with no gaps having at most two copies of each part, $17 = 1 + 1 + 3 + 5 + 7$ and $17 = 1 + 3 + 3 + 5 + 5$. The coefficient of q^{17} in the expansion of (1.3) is 2.

$$(7,5,3,1,1) \longmapsto \begin{pmatrix} 7 & 5 & 3 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \longmapsto (8,5,3,1),$$

$$(5,5,3,3,1) \longmapsto \begin{pmatrix} 9 & 5 & 1 \\ 0 & 1 & 1 \end{pmatrix} \longmapsto (9,6,2).$$

Remark 6. With the notation of the Theorem 4.1, the number of repeated parts in the partition μ equals the number of even parts in the partition λ. This fact can be easily observed in the example above.

Proof of Theorem 1.3. Theorem 1.3 (of Agarwal) follows directly from Theorem 4.1. □

For Theorem 1.4 the argument is similar and there is no need to give it in detail. The general term

$$(1+q)(1+q^3)\cdots(1+q^{2s-1})q^{1+3+5+\cdots+(2s+1)}$$

of the mock theta function $\Phi_1(q)$ of order 5 defined in (1.4) generates the partitions containing exactly one part equal to $2s+1$ and one or two parts equal to each one of the numbers 1, 3, 5, ..., $2s-1$. Hence, we now express n as

$$n = j_1 + 3j_2 + 5j_3 + \cdots + (2s-1)j_s + 1 + 3 + 5 + \cdots + (2s+1)$$

with $j_t \in \{0,1\}$, or as the sum of the elements of the matrix

$$A = \begin{pmatrix} (2s+1)+2j_2+\cdots+2j_s & \cdots & 5+2j_s & 3 & 1 \\ j_1 & \cdots & j_{s-1} & j_s & 0 \end{pmatrix}.$$

Theorem 4.2 (Agarwal) *The mock theta function of order 5* (1.4)

$$\Phi_1(q) = \sum_{n=0}^{\infty} (-q\,;q^2)_n q^{(n+1)^2}\,,$$

is the generating function for the number of two-line matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix} \tag{4.9}$$

with non-negative integer entries satisfying

$$c_{s+1} = 1\,, \quad d_{s+1} = 0; \tag{4.10}$$

$$d_t \in \{0,1\}\,, \quad \forall\, t \le s; \tag{4.11}$$

$$c_t = 2 + c_{t+1} + 2d_{t+1}\,, \quad \forall t \le s; \tag{4.12}$$

$$n = \sum c_t + \sum d_t. \tag{4.13}$$

Furthermore, there is a bijection between this set of matrices and each one of the following two sets of partitions:

(i) *partitions λ of n satisfying*

$$\lambda_s = 1; \tag{4.14}$$

$$\lambda_t - \lambda_{t+1} = \begin{cases} 2, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ are both odd;} \\ 3, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ have different parity;} \\ 4, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ are both even;} \end{cases} \tag{4.15}$$

(ii) *partitions μ of n into odd parts, with no gaps, containing at most two copies of any part and such that the largest part has multiplicity 1.*

(iii) *n-color partitions $\lambda = (\lambda_1)_{\alpha_1} + \ldots + (\lambda_s)_{\alpha_s}$ satisfying:*

- (a) *the weighted difference of consecutive parts is 0, i.e., $\lambda_t - \lambda_{t+1} - \alpha_t - \alpha_{t+1} = 0$;*
- (b) *the smallest part λ_s is 1;*
- (c) *$\alpha_t = 1$ if λ_t is odd and $\alpha_t = 0$ if λ_t is even.*

Given any matrix A, the corresponding partition λ is obtained by adding up the entries in each of the columns of A and μ is the partition into odd parts containing one (respectively, two) copies of a number $2t-1$ if $d_t = 0$ (respectively, $d_t = 1$) for any $t \le s =$ number of columns of A.

Example 6. For example, $n = 61$ has 4 partitions of this type,

$$61 = 1+1+3+3+5+5+7+7+9+9+11$$

$$61 = 1+3+5+5+7+7+9+11+13$$

$$61 = 1+3+3+5+7+9+9+11+13$$

$$61 = 1+1+3+5+7+9+11+11+13$$

Using a package like *Maple*, we find that the coefficient of q^{61} in the expansion of (1.4) is 4. The corresponding partitions and two-line matrices are listed below.

$$(11,9,9,7,7,5,5,3,3,1,1) \mapsto \begin{pmatrix} 19 & 15 & 11 & 7 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \mapsto (20,16,12,8,4,1)$$

$$(13,11,9,7,7,5,5,3,1) \mapsto \begin{pmatrix} 17 & 15 & 11 & 7 & 5 & 3 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \mapsto (17,15,12,8,5,3,1)$$

$$(13,11,9,9,7,5,3,3,1) \mapsto \begin{pmatrix} 17 & 13 & 11 & 9 & 5 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \mapsto (17,14,11,9,6,3,1)$$

$$(13,11,11,9,7,5,3,1,1) \mapsto \begin{pmatrix} 15 & 13 & 11 & 9 & 7 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \mapsto (16,13,11,9,7,4,1)$$

Remark 7. As with Theorem 4.1, with the notation of Theorem 4.2, the number of repeated parts in the partition μ equals the number of even parts in the partition λ. This fact can be easily observed in the example above.

Proof of Theorem 1.4. Theorem 1.4 of Agarwal, mentioned in the Introduction, follows directly from Theorem 4.2. □

5 A new result of Agarwal-type

In this section we obtain a new result that belongs to the same family of results of Agarwal mentioned in the Introduction (Theorem 1.1 – Theorem 1.4). Consider the mock theta function of order 8 $U_1(q)$ defined in (1.5), which is number 40 in [10]. Its general term

$$\frac{q^{(s+1)^2}(-q\,;q^2)_s}{(-q^2\,;q^4)_{s+1}} = \frac{(1+q)(1+q^3)\cdots(1+q^{2s-1})\,q^{1+3+5+\cdots+(2s+1)}}{(1+q^2)(1+q^6)\cdots(1+q^{4s+2})},$$

apart from the signs, is the generating function for partitions containing:

(i) each of the odd numbers 1, 3, 5, $\cdots$, $2s-1$ as a part, with multiplicity 1 or 2;

(ii) exactly one part equal to $2s+1$;

(iii) any number of even parts which are not multiples of 4 and are less than or equal to $4s+2$.

Concerning condition (iii) above, note that each part that is not a multiple of 4 is of the form $4t-2$ and, hence, is equivalent to two parts equal to $2t-1$. Therefore, (i), (ii), and (iii) are equivalent to

(i)′ each one of the odd numbers 1, 3, 5, $\cdots$, $2s-1$ as a part, with multiplicity at least 1;

(ii)′ $2s+1$ is a part with multiplicity odd.

Hence, it follows from Theorem 2.1 that there is a representation in terms of n-color partitions. Indeed, if we decompose n as

$$n=(1+j_1)1+(1+j_2)3+\cdots+(1+j_s)(2s-1)+(2s+1)+2e_1+6e_2+\cdots+(4s+2)e_{s+1},$$

with $e_t \geq 0$ and $j_t \in \{0,1\}$ or, equivalently, as the sum of the entries of the $2\times(s+1)$-matrix

$$A = \begin{pmatrix} (2s+1)+j_1+\sum_{t=2}^{s} 2j_t+\sum_{t=2}^{s+1} 4e_t & \cdots & 3+j_s+4e_{s+1} & 1 \\ 2e_1 & \cdots & 2e_s & 2e_{s+1} \end{pmatrix}.$$

Then, n is the sum of the entries of a matrix of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix} \tag{5.1}$$

with non-negative integer coefficients satisfying

$$c_{s+1} = 1\,; \tag{5.2}$$

$$d_t \text{ is even}\,,\ d_t \geq 0\,, \quad \forall t\,; \tag{5.3}$$

$$c_t \equiv 1, \quad c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 2 + c_{t+1} + 2d_{t+1}\,; \tag{5.4}$$

$$c_t \equiv 0, \quad c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 3 + c_{t+1} + 2d_{t+1}\,; \tag{5.5}$$

$$c_t \equiv 1, \quad c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 3 + c_{t+1} + 2d_{t+1}\,; \tag{5.6}$$

$$c_t \equiv 0, \quad c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 4 + c_{t+1} + 2d_{t+1}\,; \tag{5.7}$$

$$n = \sum c_t + \sum d_t\,. \tag{5.8}$$

Each matrix is to be counted with a weight that reflects the parity of the number of even parts in (*iii*). More precisely,

$$w = (-1)^{e_1+\cdots+e_{s+1}} = (-1)^{(d_1+\cdots+d_{s+1})/2}\,.$$

Note that adding up the elements in each column of the above matrix A, we obtain a partition $\lambda = (\lambda_1, \ldots, \lambda_{s+1})$ of n satisfying

$$\lambda_{s+1} \equiv 1 \pmod 2 \tag{5.9}$$

$$\lambda_t - \lambda_{t+1} = \begin{cases} 2 + d_t + d_{t+1}, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ are both odd;} \\ 3 + d_t + d_{t+1}, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ have different parity;} \\ 4 + d_t + d_{t+1}, & \text{if } \lambda_t \text{ and } \lambda_{t+1} \text{ are both even.} \end{cases} \tag{5.10}$$

This is equivalent to assigning to each part λ_t the subscript

$$\alpha_t = \begin{cases} 1 + d_t, & \text{if } \lambda_t \text{ is odd;} \\ 2 + d_t, & \text{if } \lambda_t \text{ is even,} \end{cases} \tag{5.11}$$

and requiring that the weighted difference of consecutive parts be 0.

We have the following results.

Theorem 5.1 *The mock theta function $U_1(q)$, given by* (1.5)*, is the generating function for the weighted matrices of the form* (5.1) *with non-negative integer coefficients satisfying* (5.2)–(5.8)*. Each matrix is counted with the weight*

$$w = (-1)^{e_1+\cdots+e_{s+1}} = (-1)^{(d_1+\cdots+d_{s+1})/2}\,.$$

Theorem 5.2 *There is a bijection between the following three sets:*

(*i*) *the set of matrices of the form* (5.1) *with non-negative integer entries satisfying* (5.2)–(5.8)*;*

(*ii*) *the set of partitions μ of n into odd parts with no gaps such that the largest part has multiplicity odd;*

(*iii*) *the set of n-color partitions of n $\lambda = (\lambda_1)_{\alpha_1} + (\lambda_2)_{\alpha_2} + \cdots + (\lambda_{s+1})_{\alpha_{s+1}}$ such that λ_{s+1} is odd, and, as a consequence of Remark 1, odd parts appear with odd subscripts and even parts with even subscripts.*

Proof. Given a matrix A of the form (5.1) with non-negative integer entries satisfying (5.2)–(5.8), the partition λ is obtained by adding up the elements in each column of A, with subscripts given by (5.11). As is Theorem 2.1, the partition μ is obtained in the following way:

– μ contains $1 + d_t$ copies of $2t - 1$;

– whenever c_t is even, μ contains an additional copy of $2t-1$.

■

Example 7. Considering the beginning of the expansion

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(-q^2\,;q^4)_{n+1}} = q - q^3 + q^4 + 2q^5 - q^6 - 2q^7 + q^8 + 3q^9 - q^{10} - 4q^{11} + 2q^{12} + 5q^{13} - 2q^{14} - 6q^{15} + 3q^{16} + 8q^{17} - 4q^{18} - 9q^{19} + 4q^{20} + 11q^{21} - 5q^{22} - 14q^{23} + \cdots$$

and of its unsigned version

$$\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q^2\,;q^4)_{n+1}} = q + q^3 + q^4 + 2q^5 + q^6 + 2q^7 + q^8 + 3q^9 + 3q^{10} + 4q^{11} + 4q^{12} + 5q^{13} + 4q^{14} + 6q^{15} + 7q^{16} + 8q^{17} + 8q^{18} + 11q^{19} + 10q^{20} + 13q^{21} + 15q^{22} + \cdots$$

we see that for $n = 20$, for example, there are 10 matrices, and the sum of their weights is 4, i.e., there are seven matrices with weight $+1$ and three with weight -1. Indeed, we can construct the following table. In the table below, for example, $(3, 1^{17})$ denotes a partition in which 1 appears seventeen times as a part and 3 appears with multiplicity one.

matrix	λ	μ	w
$\begin{pmatrix} 3 & 1 \\ 16 & 0 \end{pmatrix}$	$(19_{17}, 1_1)$	$(3, 1^{17})$	$+1$
$\begin{pmatrix} 7 & 1 \\ 10 & 2 \end{pmatrix}$	$(17_{11}, 3_3)$	$(3, 3, 3, 1^{11})$	$+1$
$\begin{pmatrix} 11 & 1 \\ 4 & 4 \end{pmatrix}$	$(15_5, 5_5)$	$(3, 3, 3, 3, 3, 1, 1, 1, 1, 1)$	$+1$
$\begin{pmatrix} 6 & 3 & 1 \\ 10 & 0 & 0 \end{pmatrix}$	$(16_{12}, 3_1, 1_1)$	$(5, 3, 1^{12})$	-1
$\begin{pmatrix} 10 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$	$(14_6, 5_3, 1_1)$	$(5, 3, 3, 3, 1, 1, 1, 1, 1, 1)$	-1
$\begin{pmatrix} 10 & 7 & 1 \\ 0 & 0 & 2 \end{pmatrix}$	$(10_2, 7_1, 3_3)$	$(5, 5, 5, 3, 1, 1)$	-1
$\begin{pmatrix} 7 & 4 & 1 \\ 8 & 0 & 0 \end{pmatrix}$	$(15_9, 4_2, 1_1)$	$(5, 3, 3, 1^9)$	$+1$
$\begin{pmatrix} 11 & 4 & 1 \\ 2 & 2 & 0 \end{pmatrix}$	$(13_3, 6_4, 1_1)$	$(5, 3, 3, 3, 3, 1, 1, 1)$	$+1$
$\begin{pmatrix} 7 & 5 & 3 & 1 \\ 4 & 0 & 0 & 0 \end{pmatrix}$	$(11_5, 5_1, 3_1, 1_1)$	$(7, 5, 3, 1, 1, 1, 1, 1)$	$+1$
$\begin{pmatrix} 10 & 6 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$(10_2, 6_2, 3_1, 1_1)$	$(7, 5, 3, 3, 1, 1)$	$+1$

It is interesting to point out that, except for a displacement $n \mapsto n+1$, the mock theta function (1.5) is a member of the 3-parameter family

$$\sum_{n=0}^{\infty} \frac{(-q^i;q^{2i})_n q^{rn^2}}{(q^j;q^{2j})_n} \,. \tag{5.12}$$

In the study of the generating function (5.12), we have to distinguish two cases, the case in which i is not a multiple of j, and the case in which it does. For the first case we have the theorem below. In the second case, if i is a multiple of j (in particular, if $i = j$), instead of being the generating function for a class of n-color partitions, (5.12) can generate pairs of n-color partitions, as we will see in the next section.

Theorem 5.3 *Let i, j, and r be positive integers and suppose that i is not a multiple of j. Then, the coefficient of q^n in the expansion of* (5.12) *is equal to the number of elements in each one of the sets*

(*i*) *the set of partitions μ of n into odd parts with no gaps and such that, for each $t \in \{1,2,3,\ldots,s\}$, the number $2t-1$ is a part of μ with a multiplicity of the form $r+jk$ or $r+i+jk$, where $k \geq 0$;*

(*ii*) *the set of n-color partitions $\lambda = (\lambda_1)_{\alpha_1} + \cdots + (\lambda_s)_{\alpha_s}$ of n satisfying*

$$\begin{aligned} \alpha_t &\in \{r+jk \mid k \geq 0\} \cup \{r+i+jk \mid k \geq 0\}, \quad \forall t\,; \\ \lambda_s &= \alpha_s\,; \\ \lambda_t - \lambda_{t+1} &= \alpha_t + \alpha_{t+1}, \quad \forall t < s. \end{aligned}$$

Theorem 5.3 can be proved using the same argument employed in the proofs of Theorems 5.1 and 5.2 above. It suffices to note that the general term

$$\frac{(1+q^i)(1+q^{3i})\cdots(1+q^{(2s-1)i})\, q^{r(1+3+\cdots+(2s-1))}}{(1-q^j)(1-q^{3j})\cdots(1-q^{(2s-1)j})}$$

of (5.12) generates the partitions μ into odd parts with no gaps such that, for any $t \in \{1,2,\ldots,s\}$, the number $2t-1$ is a part of μ with a multiplicity α_t of the form $r+kj$ or $r+i+kj$, with $k \geq 0$. Then, it suffices to take λ as the conjugate to μ with respect to the modular Ferrers diagram, taking α_t defined above as the subscript of λ_t.

The argument outlined above works only when i is not a multiple of j.

Example 8. Consider the particular case in which $i = 5$, $j = 3$, and $r = 2$ in (5.12). Using a package like *Maple*, we see that the coefficient of q^{37} in the expansion of

$$\sum_{n=0}^{\infty} \frac{(-q^5;q^{10})_n q^{2n^2}}{(q^3;q^6)_n} = 1+q^2+q^5+q^7+2q^8+\cdots+10q^{35}+5q^{36}+7q^{37}+12q^{38}+\cdots$$

is 7. This means that there are 7 partitions of 37 of each one of the two types described in Theorem 5.3. We have

$$\alpha_t \in \{2,5,8,11,\ldots\} \cup \{7,10,13,16,\ldots\} := N$$

To construct the partition μ, it suffices to express

$$37 = 1\cdot\alpha_s + 3\cdot\alpha_{s-1} + 5\cdot\alpha_{s-2} + \cdots\,, \qquad \text{with} \quad \alpha_t \in N.$$

By inspection, we construct the following table listing the 7 possibilities (the multiplicities α_t for μ are the subscripts of λ).

μ	λ
(1^{37})	(37_{37})
$(3^2, 1^{31})$	$(35_{31}, 2_2)$
$(3^5, 1^{22})$	$(32_{22}, 5_5)$
$(3^8, 1^{13})$	$(29_{13}, 8_8)$
$(3^7, 1^{16})$	$(30_{16}, 7_7)$
$(3^{10}, 1^7)$	$(27_7, 10_{10})$
$(7^2, 5^2, 3^2, 1^7)$	$(19_7, 10_2, 6_2, 2_2)$

6 Two new results of Agarwal-type for pairs of partitions

In this section, we consider an example of the generating function (5.12) in which i is a multiple of j ($i = j$, indeed) and also a variant involving a displacement $n \mapsto n + 1$. We find combinatorial interpretations in terms of pairs of n-color partitions for the mock theta functions (1.6) and (1.7). Actually, for the second element of the pair we have to slightly relax the definition. Consider the mock theta function of order 8 $V_1(q)$ defined in (1.6), which is number 43 in [10]. Its general term

$$\frac{q^{(s+1)^2}(-q\,;q^2)_s}{(q\,;q^2)_{s+1}} = \frac{(1+q)(1+q^3)\cdots(1+q^{2s-1})q^{1+3+5+\cdots+(2s+1)}}{(1-q)(1-q^3)\cdots(1-q^{2s+1})}$$

is the generating function for partitions into parts of two colors, say blue and green, containing

(i) each one of the odd numbers 1, 3, 5, $\cdots$, $2s-1$ as a blue part, with multiplicity 1 or 2 ;

(ii) exactly one blue part equal to $2s+1$;

(iii) any number of odd green parts less than or equal to $2s+1$.

Accordingly we decompose n as

$$\begin{aligned} n = {} & (1+j_1)1 + (1+j_2)3 + \cdots + (1+j_s)(2s-1) + (2s+1) \\ & + 1d_1 + 3d_2 + \cdots + (2s+1)d_{s+1}, \end{aligned}$$

with $d_t \geq 0$ and $j_t \in \{0, 1\}$ or, equivalently, as the sum of the entries of the $2 \times (s+1)$-matrix

$$A = \begin{pmatrix} (2s{+}1){+}j_1{+}\sum_{t=2}^{s} 2j_t{+}\sum_{t=2}^{s+1} 2d_t & \cdots & 3{+}j_s{+}2d_{s+1} & 1 \\ d_1 & \cdots & d_s & d_{s+1} \end{pmatrix}. \tag{6.1}$$

Then, n is the sum of entries of a matrix of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_{s+1} \\ d_1 & d_2 & \cdots & d_{s+1} \end{pmatrix} \tag{6.2}$$

with non-negative integer coefficients satisfying

$$c_{s+1} = 1\,; \tag{6.3}$$

$$d_t \geq 0\,, \quad \forall t\,; \tag{6.4}$$

$$c_t \equiv 1, \quad c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 2 + c_{t+1} + 2d_{t+1}\,; \tag{6.5}$$

$$c_t \equiv 0, \quad c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 3 + c_{t+1} + 2d_{t+1}\,; \tag{6.6}$$

$$c_t \equiv 1, \quad c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 3 + c_{t+1} + 2d_{t+1}\,; \tag{6.7}$$

$$c_t \equiv 0, \quad c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 4 + c_{t+1} + 2d_{t+1}\,; \tag{6.8}$$

$$n = \sum c_t + \sum d_t\,. \tag{6.9}$$

We have the following result.

Theorem 6.1 *The mock theta function $V_1(q)$, given by* (1.6)*, is the generating function for the matrices of the form* (6.2) *with non-negative integer entries satisfying* (6.3)–(6.9)*.*

Unfortunately, if we add up the elements in the columns of a matrix (6.2), the partition we obtain is not necessarily of the type in the theorems of Agarwal-type. But we still have an interpretation in terms of pairs of partitions of the Agarwal-type. Indeed, we may express (6.2) as

$$A \longleftrightarrow (B, C),$$

where

$$B = \begin{pmatrix} (2s+1)+\sum_{t=2}^{s} 2j_t & \cdots & 3 & 1 \\ j_1 & \cdots & j_s & 0 \end{pmatrix} \tag{6.10}$$

and

$$C = \begin{pmatrix} \sum_{t=2}^{s+1} 2d_t & \cdots & 2d_{s+1} & 0 \\ d_1 & \cdots & d_s & d_{s+1} \end{pmatrix}. \tag{6.11}$$

Then, B and C are matrices of the form

$$B = \begin{pmatrix} b_1 & \cdots & b_s & b_{s+1} \\ j_1 & \cdots & j_s & j_{s+1} \end{pmatrix} \tag{6.12}$$

and

$$C = \begin{pmatrix} f_1 & \cdots & f_s & f_{s+1} \\ d_1 & \cdots & d_s & d_{s+1} \end{pmatrix} \tag{6.13}$$

with non-negative integer entries satisfying

$$b_{s+1} = 1\,, \quad j_{s+1} = 0; \tag{6.14}$$

$$j_t \in \{0, 1\}\,, \quad \forall t\,; \tag{6.15}$$

$$b_t = 2 + b_{t+1} + 2j_{t+1}\,; \tag{6.16}$$

$$f_{s+1} = 0\,; \tag{6.17}$$

$$d_t \geq 0\,, \quad \forall t\,; \tag{6.18}$$

$$f_t = f_{t+1} + 2d_{t+1}\,; \tag{6.19}$$

$$n = \sum b_t + \sum j_t + \sum f_t + \sum d_t\,. \tag{6.20}$$

Adding up the elements of each of the columns of the matrices (6.12) and (6.13), we obtain partitions $\lambda = (\lambda_1, \ldots, \lambda_{s+1})$, $\lambda_t = b_t + j_t$, and $\pi = (\pi_1, \pi_2, \ldots)$, $\pi_t = f_t + d_t$ (we have to consider only non-vanishing π_t and, hence, it is possible to obtain the empty partition). Then, $\lambda = (\lambda_1)_{\alpha_1} + \cdots + (\lambda_{s+1})_{\alpha_{s+1}}$ and $\pi = (\pi_1)_{\beta_1} + (\pi_2)_{\beta_2} + \cdots$ are n-color partitions, with subscripts defined by

$$\alpha_t = 1 + j_t\,; \tag{6.21}$$

$$\beta_t = d_t\,. \tag{6.22}$$

Indeed,

$$\lambda_t - \lambda_{t+1} = 2 + j_t + j_{t+1} = \alpha_t + \alpha_{t+1}\,;$$

$$\pi_t - \pi_{t+1} = d_t + d_{t+1} = \beta_t + \beta_{t+1}\,.$$

Therefore, the weighted difference of consecutive parts is 0 in λ as well as in π. Also, the smallest part of λ is $\lambda_{s+1} = \alpha_{s+1}$. The smallest part of π is also equal to its subscript. We then have the theorem below establishing a bijection between the set of matrices of the form (6.2) with entries satisfying (6.3)–(6.9) and a set of pairs (λ, π) of n-color partitions. However, we need to consider as admissible, for the second element of the pair, the empty partitions and also partitions having vanishing subscript. We require that the subscript of the smallest part must be non-zero.

Theorem 6.2 *There is a bijection between the set of matrices of the form* (6.2) *with non-negative integer entries satisfying* (6.3)–(6.9) *and the set of pairs* (λ, π) *of n-color partitions such that*

(i) $\lambda = (\lambda_1)_{\alpha_1} + \cdots + (\lambda_{s+1})_{\alpha_{s+1}}$ *is an n-color partition, such that even parts have subscript 2, odd parts have subscript 1, the weighted difference of consecutive parts is 0, and the smallest part* 1_1*;*

(ii) $\pi = (\pi_1)_{\beta_1} + (\pi_2)_{\beta_2} + \cdots$ *is an n-color partition (possibly empty) such that the number of parts of* π *is less than or equal to the number of parts of* λ*, the weighted difference of consecutive parts is 0, and if* π *is non-empty the smallest part is equal to its subscript (the other parts* π_t *may have a subscript* $\beta_t = 0$*, but we still require* $0 \le \beta_t \le \pi_t$*);*

(iii) $n = |\lambda| + |\mu|$.

If A is a matrix (6.2) *with non-negative integer entries satisfying* (6.3)–(6.9)*, then the parts of* λ *are*

$$\lambda_t = c_t - (2d_{t+1} + \cdots + 2d_{s+1}), \quad \forall t \le s;$$
$$\lambda_{s+1} = 1,$$

with subscripts given by

$$\alpha_t = \begin{cases} 1, & \text{if } c_t \text{ is odd;} \\ 2, & \text{if } c_t \text{ is even.} \end{cases}$$

The parts of π *are the non-vanishing elements in the list*

$$\pi_t = d_t + 2d_{t+1} + \cdots + 2d_{s+1}, \quad \forall t \le s;$$
$$\pi_{s+1} = d_{s+1},$$

with subscripts given by

$$\beta_t = d_t .$$

Example 9. Considering the beginning of the expansion

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q\,;q^2)_n}{(q\,;q^2)_{n+1}} = q+q^2+q^3+2q^4+3q^5+3q^6+4q^7+5q^8+6q^9+8q^{10}+9q^{11} +11q^{12}+14q^{13}+16q^{14}+19q^{15}+23q^{16}+27q^{17}+\cdots$$

we see that for $n = 13$, for example, there are 14 matrices. The following table shows the correspondence between matrices and pairs of partitions in this particular case.

matrix	λ	π	matrix	λ	π
$\begin{pmatrix} 1 \\ 12 \end{pmatrix}$	(1_1)	(12_{12})	$\begin{pmatrix} 3 & 1 \\ 9 & 0 \end{pmatrix}$	$(3_1, 1_1)$	(9_9)
$\begin{pmatrix} 4 & 1 \\ 8 & 0 \end{pmatrix}$	$(4_2, 1_1)$	(8_8)	$\begin{pmatrix} 5 & 1 \\ 6 & 1 \end{pmatrix}$	$(3_1, 1_1)$	$(8_6, 1_1)$
$\begin{pmatrix} 6 & 1 \\ 5 & 1 \end{pmatrix}$	$(4_2, 1_1)$	$(7_5, 1_1)$	$\begin{pmatrix} 7 & 1 \\ 3 & 2 \end{pmatrix}$	$(3_1, 1_1)$	$(7_3, 2_2)$
$\begin{pmatrix} 8 & 1 \\ 2 & 2 \end{pmatrix}$	$(4_2, 1_1)$	$(6_2, 2_2)$	$\begin{pmatrix} 9 & 1 \\ 0 & 3 \end{pmatrix}$	$(3_1, 1_1)$	$(6_0, 3_3)$
$\begin{pmatrix} 5 & 3 & 1 \\ 4 & 0 & 0 \end{pmatrix}$	$(5_1, 3_1, 1_1)$	(4_4)	$\begin{pmatrix} 6 & 3 & 1 \\ 3 & 0 & 0 \end{pmatrix}$	$(6_2, 3_1, 1_1)$	(3_3)
$\begin{pmatrix} 7 & 4 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$(7_1, 4_2, 1_1)$	(1_1)	$\begin{pmatrix} 8 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$(8_2, 4_2, 1_1)$	$\emptyset$
$\begin{pmatrix} 7 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$(6_2, 3_1, 1_1)$	$(3_1, 1_1)$	$\begin{pmatrix} 8 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$(6_2, 3_1, 1_1)$	$(2_0, 1_1)$

A similar result holds for the mock theta function $V_0(q)$, given by (1.7) (Number 41 in [10]). It suffices to examine the series

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(-q\,;q^2)_n}{(q\,;q^2)_n}\,. \tag{6.23}$$

By looking at its general term

$$\frac{q^{s^2}(-q\,;q^2)_s}{(q\,;q^2)_s} = \frac{(1+q)(1+q^3)\cdots(1+q^{2s-1})\,q^{1+3+5+\cdots+(2s-1)}}{(1+q)(1+q^3)\cdots(1+q^{2s-1})}$$

it is easy to see that the coefficient of q^n counts the number of ways to express n as the sum of the entries of the $(2\times s)-$matrix

$$A=\begin{pmatrix} (2s-1)+j_1+\sum_{t=2}^{s}2j_t+\sum_{t=2}^{s}2d_t & \cdots & 3+j_{s-1}+2j_s+2d_s & 1+j_s \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}, \tag{6.24}$$

with $d_t \geq 0$ and $j_t \in \{0,1\}$. As in the previous case, we have the following result.

Theorem 6.3 *The series* (6.23) *is the generating function for the matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{6.25}$$

with non-negative integer coefficients satisfying

$$c_s \in \{1,2\}\,; \tag{6.26}$$
$$d_t \geq 0\,, \quad \forall t\,; \tag{6.27}$$
$$c_t \equiv 1, \quad c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 2 + c_{t+1} + 2d_{t+1}\,; \tag{6.28}$$
$$c_t \equiv 0, \quad c_{t+1} \equiv 1 \pmod 2 \Longrightarrow c_t = 3 + c_{t+1} + 2d_{t+1}\,; \tag{6.29}$$
$$c_t \equiv 1, \quad c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 3 + c_{t+1} + 2d_{t+1}\,; \tag{6.30}$$
$$c_t \equiv 0, \quad c_{t+1} \equiv 0 \pmod 2 \Longrightarrow c_t = 4 + c_{t+1} + 2d_{t+1}\,; \tag{6.31}$$
$$n = \sum c_t + \sum d_t\,. \tag{6.32}$$

As in the case of $V_1(q)$ considered before, we consider a correspondence

$$A \longleftrightarrow (B,C),$$

where

$$B = \begin{pmatrix} (2s-1)+\sum_{t=2}^{s}2j_t & \cdots & 3+2j_s & 1 \\ j_1 & \cdots & j_{s-1} & j_s \end{pmatrix} \tag{6.33}$$

and

$$C = \begin{pmatrix} \sum_{t=2}^{s}2d_t & \cdots & 2d_s & 0 \\ d_1 & \cdots & d_{s-1} & d_s \end{pmatrix}. \tag{6.34}$$

By exactly the same argument as before, we obtain the following theorem.

Theorem 6.4 *There is a bijection between the set of matrices of the form* (6.25) *with non-negative integer entries satisfying* (6.26)–(6.32) *and the set of pairs* (λ,π) *of* n*-color partitions such that*

(*i*) $\lambda = (\lambda_1)_{\alpha_1} + \cdots + (\lambda_{s+1})_{\alpha_{s+1}}$ *is an* n*-color partition, such that even parts have subscript 2, odd parts have subscript 1, the weighted difference of consecutive parts is 0, and the smallest part is* 1_1 *or* 2_2*;*

(ii) $\pi = (\pi_1)_{\beta_1} + (\pi_2)_{\beta_2} + \cdots$ *is an n-color partition (possibly empty) such that the number of parts of* π *is less than or equal to the number of parts of* λ*, the weighted difference of consecutive parts is 0, and if* π *is non-empty the smallest part is equal to its subscript (the other parts* π_t *may have a subscript* $\beta_t = 0$*, but we still require* $0 \le \beta_t \le \pi_t$*);*

(iii) $n = |\lambda| + |\mu|$.

If A is a matrix (6.25) *with non-negative integer entries satisfying* (6.26)–(6.32)*, then the parts of* λ *are*

$$\begin{aligned} \lambda_t &= c_t - (2d_{t+1} + \cdots + 2d_s)\,, \quad \text{if } t < s; \\ \lambda_s &= c_s, \end{aligned}$$

with subscripts given by

$$\alpha_t = \begin{cases} 1, & \text{if } c_t \text{ is odd;} \\ 2, & \text{if } c_t \text{ is even.} \end{cases}$$

The parts of π *are the non-vanishing elements in the list*

$$\begin{aligned} \pi_t &= d_t + 2d_{t+1} + \cdots + 2d_s\,, \quad (t < s); \\ \pi_s &= d_s, \end{aligned}$$

with subscripts given by

$$\beta_t = d_t\,.$$

Remark 8. Looking at the results of this and the previous sections, we realize that when we have an interpretation in terms of two-line matrices, the occurrence of 2 in a condition of the form

$$c_t = b + c_{t+1} + 2d_{t+1} \tag{6.35}$$

is an indication that an interpretation in terms of n-color partitions might be possible. Indeed, taking $\lambda_t = c_t + d_t$ and $\alpha_t = b/2 + d_t$, condition (6.35) can be restated as $\lambda_t - \lambda_{t+1} - \alpha_t - \alpha_{t+1} = 0$.

7 A further generalization

In this section we consider the family of generating functions of the form

$$\sum_{n=0}^{\infty} \frac{(-q^i; q^k)_n q^{rn^2+an}}{(q^j\,; q^\ell)_n}\,,$$

which includes as particular cases some of the generating functions considered before, for example (5.12). In what follows we consider $2 \times s$ matrices in which the entries in the second row are integers and the entries in the first row are linear polynomials (polynomials of degree less than or equal to 1) with integer coefficients. Of course, this is equivalent to using matrices with three rows.

Theorem 7.1 *For fixed integers* $i, k, j, \ell, r \ge 1$ *and* $a \ge 0$*, consider the generating function*

$$\sum_{n=0}^{\infty} \frac{(-q^i; q^k)_n q^{rn^2+an}}{(q^j\,; q^\ell)_n}\,. \tag{7.1}$$

Then, the coefficient of q^n *in the expansion of* (7.1) *is the number of matrices of the form*

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{7.2}$$

with non-negative integer entries satisfying $A = A(1)$*, where*

$$A(x) = \begin{pmatrix} c_1(x) & c_2(x) & \cdots & c_s(x) \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{7.3}$$

is a matrix such that the entries in the first row are linear polynomials with integer coefficients and the elements in the second row are non-negative integers satisfying

$$c_s(x) = (a+r)x + i\epsilon_s\,; \tag{7.4}$$

$$d_t \geq 0 \;\; \textit{are multiples of } j\,; \tag{7.5}$$

$$c_t(x) = c_{t+1}(x) + \Big(2r + c_{t+1}(0)\,\frac{k}{i} + \frac{\ell}{j}\,d_{t+1}\Big)x + (\epsilon_t - \epsilon_{t+1})\,i\,, \forall t < s\,; \tag{7.6}$$

$$\epsilon_t \in \{0,1\}, \quad \forall t; \tag{7.7}$$

$$n = \sum c_t(1) + \sum d_t\,. \tag{7.8}$$

Proof. For any s, the general term of (7.1),

$$\frac{(1+q^i)(1+q^{i+k})\cdots(1+q^{i+(s-1)k})}{(1-q^j)(1-q^{j+\ell})\cdots(1-q^{j+(s-1)\ell})}\cdot q^{r(1+3+5+\cdots+(2s-1))}q^{a(1+1+1+\cdots+1)}\,,$$

is the generating function for partitions into parts of four colors, say blue, green, yellow, and black, containing

- (i) exactly r blue parts equal to each one of the odd numbers 1, 3, 5, ... , $2s-1$;
- (ii) distinct green parts from i, $i+k$, $i+2k$, $\cdots$, $i+(s-1)k$, i.e., congruent to i modulus k, and less than or equal to $i+(s-1)k$;
- (iii) any number of yellow parts belonging to the set $\{j, j+\ell, j+2\ell, \ldots, j+(s-1)\ell\}$;
- (iv) exactly s black parts equal to a.

Accordingly, we decompose n as

$$\begin{aligned} n = {} & r\big(1+3+5+\cdots+(2s-1)\big) + as + \Big(i\epsilon_1 + (i+k)\epsilon_2 + \cdots + (i+(s-1)k)\epsilon_s\Big) \\ & + \big(je_1 + (j+\ell)e_2 + \cdots + (j+(s-1)\ell)e_s\big)\,, \end{aligned}$$

with $\epsilon_t \in \{0,1\}$ and $e_t \geq 0$, or, equivalently, as the sum of the entries of the matrix

$$A = \begin{pmatrix} u_1 & u_2 & \cdots & u_s \\ je_1 & je_2 & \cdots & je_s \end{pmatrix}, \tag{7.9}$$

where

$$\begin{aligned} u_1 &= a + (2s-1)r + i\epsilon_1 + k\epsilon_2 + \cdots + k\epsilon_s + \ell e_2 + \cdots + \ell e_s \\ u_2 &= a + (2s-3)r + i\epsilon_2 + k\epsilon_3 + \cdots + k\epsilon_s + \ell e_3 + \cdots + \ell e_s \\ &\;\vdots \\ u_{s-1} &= a + 3r + i\epsilon_{s-1} + k\epsilon_s + \ell e_s \\ u_s &= a + r + i\epsilon_s. \end{aligned}$$

Taking $d_t = je_t$ we define a matrix

$$A(x) = \begin{pmatrix} c_1(x) & c_2(x) & \cdots & c_s(x) \\ d_1 & d_2 & \cdots & d_s \end{pmatrix}, \tag{7.10}$$

where the elements in the first row are the linear polynomials

$$c_s(x) = (a+r)x + i\epsilon_s\,;$$
$$c_t(x) = \big(a + (1+2(s-t))r + k\epsilon_{t+1} + \cdots + k\epsilon_s + \ell e_{t+1} + \cdots + \ell e_s\big)x + i\epsilon_t\,, \quad \forall t < s.$$

Then, the matrix A in (7.9) satisfies

$$A = A(1),$$

with $A(x)$ as in (7.10). Also, (7.10) is characterized by

$$c_s(x) = (a+r)x + i\epsilon_s\,;$$
$$d_t \geq 0 \text{ are multiples of } j\,;$$
$$c_t(x) = c_{t+1}(x) + \Big(2r + c_{t+1}(0)\cdot\frac{k}{i} + \frac{\ell}{j}\cdot d_{t+1}\Big)x + (\epsilon_t - \epsilon_{t+1})\,i\,, \quad \forall t < s\,;$$
$$\epsilon_t \in \{0,1\}, \quad \forall t;$$
$$n = \sum c_t(1) + \sum d_t\,.$$

The proof is complete. ■

We now consider two important particular cases of the generating function (7.1). First, for $k = i$ we have the following result.

Corollary 7.1 *For fixed integers $i, j, \ell, r \geq 1$ and $a \geq 0$, consider the generating function*

$$\sum_{n=0}^{\infty} \frac{(-q^i;q^i)_n q^{rn^2+an}}{(q^j\,;q^\ell)_n}\,. \tag{7.11}$$

Then, the coefficient of q^n in the expansion of (7.11) is the number of matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{7.12}$$

with non-negative integer entries satisfying

$$c_s = a + r + i\epsilon_s\,; \tag{7.13}$$
$$d_t \geq 0 \text{ are multiples of } j\,; \tag{7.14}$$
$$c_t = 2r + i\epsilon_t + c_{t+1} + \frac{\ell}{j}\,d_{t+1}\,, \quad \forall t < s\,; \tag{7.15}$$
$$\epsilon_t \in \{0,1\}, \quad \forall t; \tag{7.16}$$
$$n = \sum c_t + \sum d_t\,. \tag{7.17}$$

Proof. It is easy to see that $c_t(0) = i\epsilon_t$ for any t, since it holds for $t = s$ and, if it holds for some t, it also holds for $t-1$. Then, from (7.6) with $k = i$, it follows that

$$c_t(1) = 2r + i\epsilon_t + c_{t+1}(1) + \frac{\ell}{j}\,d_{t+1}\,.$$

Now, the conclusion follows easily. ■

For the case $k = 2i$, we have the following result, which can be proved by the same type of argument as above.

Corollary 7.2 *For fixed integers $i, j, \ell, r \geq 1$ and $a \geq 0$, consider the generating function*

$$\sum_{n=0}^{\infty} \frac{(-q^i; q^{2i})_n q^{rn^2+an}}{(q^j\,; q^\ell)_n}\,. \tag{7.18}$$

Then, for any n, the coefficient of q^n in the expansion of (7.18) is the number of matrices of the form

$$A = \begin{pmatrix} c_1 & c_2 & \cdots & c_s \\ d_1 & d_2 & \cdots & d_s \end{pmatrix} \tag{7.19}$$

with non-negative integer entries satisfying

$$c_s = a + r + i\epsilon_s\,; \tag{7.20}$$

$$d_t \geq 0 \ \text{ are multiples of } j\,; \tag{7.21}$$

$$c_t = 2r + i\epsilon_t + i\epsilon_{t+1} + c_{t+1} + \frac{\ell}{j}\, d_{t+1}\,, \quad \forall t < s\,; \tag{7.22}$$

$$\epsilon_t \in \{0,1\}, \quad \forall t; \tag{7.23}$$

$$n = \sum c_t + \sum d_t\,. \tag{7.24}$$

We now consider the signed version

$$\sum_{n=0}^{\infty} \frac{(q^i; q^i)_n q^{rn^2+an}}{(q^j\,; q^\ell)_n}\,, \tag{7.25}$$

of (7.11) in Corollary 7.1. It is immediate to see that now each matrix is to be counted with a weight $(-1)^{\gamma}$, where γ is the number of ϵ_t's that are equal to 1. Then, the weight of a matrix is

$$w = (-1)^{\epsilon_1 + \cdots + \epsilon_s}.$$

Looking at the element in the upper left corner of (7.9), we see that

$$c(1) - a - (2s-1)r - \frac{\ell}{j}\,(d_2 + \cdots + d_s) = i(\epsilon_1 + \cdots + \epsilon_s)\,, \qquad \text{if} \quad k = i.$$

We have the following result.

Proposition 7.1 *For fixed integers $i, j, \ell, r \geq 1$ and $a \geq 0$, the generating function*

$$\sum_{n=0}^{\infty} \frac{(q^i; q^i)_n q^{rn^2+an}}{(q^j\,; q^\ell)_n} \tag{7.26}$$

counts the number of matrices of the form (7.12) with non-negative integer coefficients satisfying conditions (7.13)–(7.17), where each matrix is to be counted with the weight

$$w = (-1)^{\frac{1}{i}\left(c_1 + a + r + \frac{\ell}{j}(d_2 + \cdots + d_s)\right)}.$$

The general case of the signed version of (7.1) in Theorem 18 can be treated by the same type of argument.

Application 2. Our goal is to find a matrix interpretation for the mock theta function of order 6

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{(q\,; q)_n q^{n^2}}{(q^3\,; q^3)_n}\,, \tag{7.27}$$

which is number 31 in [10]. This series has positive and negative integer coefficients. We start with the unsigned version

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3\,;q^3)_n}\,. \tag{7.28}$$

Applying Corollary 7.1 with $i = 1$, $j = \ell = 3$, $r = 1$, and $a = 0$, we obtain the following result.

Proposition 7.2 *The coefficient of q^n in the expansion of the generating function* (7.28)

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q^3\,;q^3)_n}$$

counts the number of matrices of the form (7.12) *with non-negative integer entries satisfying*

$$c_s \in \{1, 2\}\,; \tag{7.29}$$

$$d_t \geq 0, \quad d_t \equiv 0 \pmod 3\,, \quad \forall t\,; \tag{7.30}$$

$$c_t = 2 + c_{t+1} + d_{t+1} \qquad \textit{or} \qquad c_t = 3 + c_{t+1} + d_{t+1}\,, \quad \forall t < s\,; \tag{7.31}$$

$$n = \sum c_t + \sum d_t\,. \tag{7.32}$$

Remark 9. Note that the general term of (7.28)

$$\frac{(1+q)(1+q^2)\cdots(1+q^s)}{(1-q^3)(1-q^6)\cdots(1-q^{3s})}\, q^{1+3+5+\cdots+(2s-1)}$$

generates the two-color partitions of n containing

(i) one blue part equal to each of the numbers 1, 3, 5, ... , $2s-1$;

(ii) each one of the numbers 1, 2, 3, ... , s is a green part with multiplicity (possibly 0) $\not\equiv 2 \pmod 3$.

Remark 10. There is a bijection between the set of two-line matrices of the form (7.12) with non-negative integer entries satisfying (7.29)–(7.32) and the partitions $\lambda = (\lambda_1, \ldots, \lambda_s)$ of n into distinct parts satisfying

$$\lambda_s \equiv 1 \text{ or } 2 \pmod 3\,; \tag{7.33}$$

$$\lambda_t - \lambda_{t+1} \equiv 0 \text{ or } 2 \pmod 3, \quad \forall\, t. \tag{7.34}$$

The partition λ corresponding to a given matrix A is obtained adding up the elements in each of the columns of A.

The next result is a particular case of Proposition 7.1.

Corollary 7.3 *The mock theta function of order 6* (7.27)

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{(q\,;q)_n q^{n^2}}{(q^3\,;q^3)_n}\,,$$

which is number 31 in [10]*, is the generating function of the weighted number of matrices of the form* (7.12) *satisfying* (7.29)*–*(7.32)*. Each matrix is to be counted with the weight*

$$w = (-1)^{1+c_1+\sum_{t=2}^{s} d_t}\,.$$

Equivalently (7.27) *counts the weighted number of partitions into distinct parts satisfying* (7.33) *and* (7.34)*, where each partition $\lambda = (\lambda_1, \ldots, \lambda_s)$ is to be counted with a weight $w = (-1)^{\gamma}$, where $\gamma = \epsilon_1 + \cdots + \epsilon_s$, with*

$$\epsilon_t = \begin{cases} 0, & \text{if } \lambda_t - \lambda_{t+1} \equiv 2 \pmod 3 \\ 1, & \text{if } \lambda_t - \lambda_{t+1} \equiv 0 \pmod 3 \end{cases}\,, \quad \text{for} \quad t < s\,;$$

$$\epsilon_s = \begin{cases} 0, & \text{if } \lambda_s \equiv 1 \pmod 3 \\ 1, & \text{if } \lambda_s \equiv 2 \pmod 3 \end{cases}\,.$$

Example 10. Consider the first few terms of the expansion

$$\gamma(q) = 1+q-q^2+2q^4-2q^5-q^6+3q^7-2q^8+3q^{10}-4q^{11}-q^{12}+5q^{13}-3q^{14} -q^{15}+6q^{16}-6q^{17}-2q^{18}+7q^{19}-6q^{20}+9q^{22}-8q^{23}-3q^{24}+\cdots$$

and its unsigned version

$$\sum_{n=0}^{\infty} \frac{(-q)_n q^{n^2}}{(q^3\,;q^3)_n} = 1+q+q^2+2q^4+2q^5+q^6+3q^7+2q^8+2q^9+5q^{10}+4q^{11}+5q^{12}+ +7q^{13}+5q^{14}+7q^{15}+10q^{16}+8q^{17}+12q^{18}+15q^{19}+12q^{20}+ +18q^{21}+21q^{22}+18q^{23}+25q^{24}+\cdots$$

For $n = 15$, for example, the above expansions tell us that we have 7 matrices and that the sum of their weights is -1. In other words, out of 7 matrices, four have weight -1 and three have weight $+1$. We can construct the following table.

matrix	$1+c_1+\sum_{t=2}^{s} d_t$	$w = (-1)^{1+c_1+\sum_{t=2}^{s} d_t}$
$\begin{pmatrix} 4 & 2 \\ 9 & 0 \end{pmatrix}$	$1+4+0=5$	-1
$\begin{pmatrix} 7 & 2 \\ 3 & 3 \end{pmatrix}$	$1+7+3=11$	-1
$\begin{pmatrix} 5 & 3 & 1 \\ 6 & 0 & 0 \end{pmatrix}$	$1+5+0=6$	$+1$
$\begin{pmatrix} 8 & 3 & 1 \\ 0 & 3 & 0 \end{pmatrix}$	$1+8+3=12$	$+1$
$\begin{pmatrix} 7 & 4 & 1 \\ 3 & 0 & 0 \end{pmatrix}$	$1+7+0=8$	$+1$
$\begin{pmatrix} 6 & 4 & 2 \\ 3 & 0 & 0 \end{pmatrix}$	$1+6+0=7$	-1
$\begin{pmatrix} 8 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	$1+8+0=9$	-1

The sum of all weights is indeed -1.

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