AN IDENTITY OF ANDREWS AND A NEW METHOD FOR THE RIORDAN ARRAY PROOF OF COMBINATORIAL IDENTITIES

Eduardo H. M. Brietzke
Instituto de Matemática – UFRGS
Caixa Postal 15080
91509–900 Porto Alegre, RS, Brazil
email: brietzke@mat.ufrgs.br

Abstract

We consider an identity relating Fibonacci numbers to Pascal's triangle discovered by G. E. Andrews. Several authors provided proofs of this identity, most of them rather involved or else relying on sophisticated number theoretical arguments. We present a new proof, quite simple and based on a Riordan array argument. The main point of the proof is the construction of a new Riordan array from a given Riordan array, by the elimination of elements. We extend the method and as an application we obtain other identities, some of which are new. An important feature of our construction is that it establishes a nice connection between the generating function of the $A-$sequence of a certain class of Riordan arrays and hypergeometric functions.

1 Introduction

In this article we provide a new proof of an identity of Andrews, based on Riordan arrays. Several authors have already proved this identity using different types of argument (references are given below). Our reason to give a further proof is that we believe our idea is new and interesting on its own.

In our approach we establish a nice connection between the generating function of the $A-$sequence of a certain class of Riordan arrays and hypergeometric functions. This new connection with hypergeometric functions is probably in itself interesting and is one of the main features of this work. Our method involves constructing a new Riordan array from a given Riordan array by eliminating entire rows and parts of the remaining rows. In the proof of the identity of Andrews, this construction is applied to Pascal's triangle, but for the sake of illustrating the usefulness of our method, we make additional applications to Pascal's triangle as well as to other Riordan arrays, for example Catalan's triangle, obtaining a few more identities.

As a generalization of Pascal's, Catalan's, Motzkin's, and other triangles, D. G. Rogers introduced in 1978 ([16]) the concept of renewal array, which was further generalized to Riordan array by Shapiro et al. in 1991 ([19]). Among other applications Riordan arrays turned out to be an extremely powerful tool in dealing with combinatorial identities. R. Sprugnoli in [20] used Riordan arrays to find several combinatorial sums in closed form and also to determine their asymptotic value. For additional applications of Riordan arrays to the evaluation in closed form of sums involving binomial, Stirling, Bernoulli, and harmonic numbers, see [23]. The Riordan array technique has also been employed to show that two combinatorial sums are equivalent, regardless of whether they have a closed form expression or not (see [24]). An important problem that has occupied mathematicians for a long time is the inversion of combinatorial sums (see [15]). The concept of Riordan array provided a powerful tool to prove a large class of inversions (see, for example, [4] and [22]).

The paper is organized as follows. In the introduction we recall some basic results needed in the sequel. In Section 2 we develop our method of extracting new Riordan arrays from a given one. We also establish a connection between a certain class of Riordan arrays and hypergeometric functions. The ideas of Section 2 are applied in Section 3 to give a new proof of the identity of Andrews. Section 4 is devoted to additional applications of our method. As an illustration of our ideas, further identities are obtained. Identities of this type can often be proved directly, using generating functions and Lagrange's Inversion Formula. To show how this can be done, in Section 5 we give a direct proof of one of the identities obtained previously.

We begin by recalling Lagrange's Inversion Theorem, which is an important element needed in our study. Several forms of Lagrange's Inversion Formula exist (see [14]). We summarize some of them below.

Theorem 1.1 (Lagrange's Inversion Theorem [14]) *Suppose that a formal power series $w = w(t)$ is implicitly defined by the relation $w = t\phi(w)$, where $\phi(t)$ is a formal power series such that $\phi(0) \neq 0$. Then,*

$$[t^n]\big(w(t)\big)^k = \frac{k}{n}\,[t^{n-k}]\big(\phi(t)\big)^n. \tag{1}$$

Equivalently, for any formal power series $F(t)$,

$$[t^n]F(w(t)) = \frac{1}{n}[t^{n-1}]F'(t)\big(\phi(t)\big)^n. \tag{2}$$

In terms of generating functions,

$$\mathcal{G}\big([t^n]F(t)(\phi(t))^n\big) = \left[\frac{F(w)}{1 - t\phi'(w)}\,\middle|\, w = t\phi(w)\right]. \tag{3}$$

The above notation, i.e., $[f(w)|w = g(t)]$, means replacing w by $g(t)$ in $f(w)$, and given any sequence (b_n), $\mathcal{G}(b_n)$ stands for its generating function $\mathcal{G}(b_n) = \sum_{n=0}^{\infty} b_n t^n$.

A Riordan array is an infinite lower triangular array $D = \{d_{n,k}\}_{n,k\geq 0}$ defined by a pair of formal power series $D = \big(d(t), h(t)\big)$, for which

$$d_{n,k} = [t^n]\, d(t)\big(th(t)\big)^k, \quad \forall n, k \geq 0. \tag{4}$$

Here $[t^n]g(t)$ denotes the coefficient of t^n in $g(t)$. Pascal's triangle is an example of a Riordan array. In this case, $d(t) = h(t) = 1/(1-t)$ and $d_{n,k} = \binom{n}{k}$. One of the main results of the theory of Riordan arrays is the following theorem.

Theorem 1.2 ([20],Theorem 1.1) *Let $D = \big(d(t), h(t)\big)$ be a Riordan array and $f(t) = \sum f_k t^k$ a formal power series. Then,*

$$\sum_{k=0}^{\infty} f_k d_{n,k} = [t^n]\, d(t) f\big(th(t)\big). \tag{5}$$

A Riordan array $D = \big(d(t), h(t)\big)$ is called proper if $h_0 = h(0) \neq 0$. In [16], D. Rogers pointed out that proper Riordan arrays can be alternately characterized by a pair $d(t) = \sum_n d_{n,0} t^n$, the generating function of the first column, and $A(t) = \sum a_k t^k$, the generating function of the $A-$sequence, such that

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots, \quad \forall n, k \geq 0. \tag{6}$$

If $D = \big(d(t), h(t)\big)$ is a proper Riordan array, then $\mathrm{ord}\big((th(t))^k\big) = k$, for every k, where for any non-zero formal power series $g(t)$ the order $\mathrm{ord}\big(g(t)\big)$ is the index of the first non-zero coefficient of $g(t)$. Therefore, there exists a unique sequence (a_k), called the $A-$sequence of the Riordan array, such that

$$h(t) = a_0 + a_1 th(t) + a_2\big(th(t)\big)^2 + \cdots$$

[i.e., the formal power series $A(t) = a_0 + a_1 t + a_2 t^2 + \cdots$, refered to as the generating function of the $A-$sequence, is such that $h(t) = A\big(th(t)\big)$]. Multiplying by $d(t)\big(th(t)\big)^k$, we obtain

$$t^{-1}d(t)(th(t))^{k+1} = a_0 d(t)(th(t))^k + a_1 d(t)(th(t))^{k+1} + a_2 d(t)(th(t))^{k+2} + \cdots$$

and applying $[t^n]$ to both sides, (6) follows.

The converse is also true and we state it as the following theorem.

Theorem 1.3 ([20], Theorem 1.3) *Let $D = \{d_{n,k}\}_{n\geq k\geq 0}$ be an infinite triangle such that $d_{0,0} \neq 0$ and for which* (6) *holds for some sequence (a_k) with $a_0 \neq 0$. Then D is a proper Riordan array $\big(d(t), h(t)\big)$, where $d(t) = \sum_{n=0}^{\infty} d_{n,0}t^n$ is the generating function of the first column and $h(t)$ is the unique solution of*

$$h(t) = A\big(th(t)\big), \tag{7}$$

for $A(t) = \mathcal{G}(a_k) = \sum a_k t^k$ the generating funtion of the sequence (a_k). Moreover,

$$[t^{n-1}]h(t) = \frac{1}{n}[t^{n-1}]\big(A(t)\big)^n. \tag{8}$$

Proof. Since $a_0 \neq 0$, by Lagrange's Inversion Formula (1) with $w := th(t)$, $k = 1$ and $\phi = A$, (7) defines a unique formal power series $h(t)$, for which (8) holds. We have to verify that given n,

$$d_{n,k} = [t^n]d(t)\big(th(t)\big)^k,$$

for all k. By induction, suppose this holds for some n. Then,

$$\begin{aligned} d_{n+1,k+1} &= \sum_{j\geq 0} a_j d_{n,k+j} = \sum_{j\geq 0} a_j [t^n]d(t)\big(th(t)\big)^{k+j} = [t^n]d(t)\big(th(t)\big)^k A\big(th(t)\big) \\ &= [t^n]d(t)\big(th(t)\big)^k h(t) = [t^{n+1}]d(t)\big(th(t)\big)^{k+1}. \qquad \square \end{aligned}$$

2 Construction of a new Riordan array

We now describe a process of obtaining new Riordan arrays from a given Riordan array, which corresponds to eliminating rows from the original array, eliminating the first elements from the remaining rows, and shifting them to the left. For a fixed p we keep one of every p rows.

Theorem 2.1 *Given a proper Riordan array $\{d_{n,k}\}_{n,k\geq 0}$, for any integers $p \geq 2$ and $r \geq 0$, $\tilde{d}_{n,k} = d_{pn+r,(p-1)n+r+k}$ $(n, k \geq 0)$ defines a new Riordan array. Moreover, the generating function of the $A-$sequence of the new array is $\big(A(t)\big)^p$, where $A(t)$ is the generating function of the $A-$sequence of the given Riordan array.*

Proof. Let $A(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ be as in (6). If $p = 2$, for $r = 0, 1$ we have

$$\begin{aligned}\tilde{d}_{n+1,k+1} = d_{2n+2+r,\, n+k+2+r} &= \sum_{i=0}^{\infty} a_i d_{2n+1+r,\, n+k+i+1+r} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i a_j d_{2n+r,\, n+k+i+j+r}\end{aligned}$$

and, therefore,

$$\tilde{d}_{n+1,k+1} = \sum_{\nu=0}^{\infty}\sum_{i=0}^{\nu} a_i a_{r-i} d_{2n+r,n+k+\nu+r},$$

i.e.,

$$\tilde{d}_{n+1,k+1} = \sum_{\nu=0}^{\infty}\sum_{i=0}^{\nu} a_i a_{r-i} \tilde{d}_{n,k+\nu}.$$

Hence,

$$\tilde{d}_{n+1,k+1} = \sum_{r=0}^{\infty} b_r \tilde{d}_{n,k+r}, \qquad \text{where} \quad \sum_{k=0}^{\infty} b_k t^k = \big(A(t)\big)^2.$$

By Theorem **??**, $\{\tilde{d}_{n,k}\}_{n,k\geq 0}$ is a Riordan array and $B(t) = \big(A(t)\big)^2$ is the generating function of its $A-$sequence. If $p \geq 3$ an iteration of the argument applies. $\square$

For example, beginning with Pascal's triangle, for $p = 3$ and $r = 1$, we obtain the Riordan array

1						
4	1					
21	7	1				
120	45	10	1			
715	286	78	13	1		
2002	1001	560	120	16	1	
27132	11628	3876	969	171	19	1
		······				

in which $d_{n,k} = \binom{3n+1}{2n+k+1}$ and, by Theorem 2.1,

$$d_{n+1,k+1} = d_{n,k} + 3d_{n,k+1} + 3d_{n,k+2} + d_{n,k+3}.$$

In what follows we use the generalized hypergeometric series, defined by

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ c_1, \ldots, c_q \end{array} \middle| \, t \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(c_1)_n \cdots (c_q)_n} \cdot \frac{t^n}{n!},$$

where $(a)_n$ stands for the Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\cdots(a+n-1), & \text{if } n \geq 1 \end{cases}$$

The hypergeometric series is characterized by the fact that its constant term is 1 and, setting $A_n = \frac{(a_1)_n \cdots (a_p)_n}{(c_1)_n \cdots (c_q)_n}\, t^n$, the ratio of consecutive terms is

$$\frac{A_{n+1}}{A_n} = \frac{(a_1+n)\cdots(a_p+n)}{(c_1+n)\cdots(c_q+n)} \cdot \frac{t}{n+1}.$$

In order to apply Theorem 2.1 we need the following result, which establishes an interesting connection between Riordan arrays and hypergeometric functions.

Theorem 2.2 *If the generating function of the $A-$sequence of a proper Riordan array is $A(t) = (1+t)^q$, with $q \in \mathbb{N}$, then h is the hypergeometric function*

$$h(t) = {}_qF_{q-1}\left(\begin{array}{c} \frac{q}{q}, \frac{q+1}{q}, \ldots, \frac{2q-1}{q} \\ \frac{q+1}{q-1}, \frac{q+2}{q-1}, \ldots, \frac{2q-1}{q-1} \end{array} \middle| \frac{q^q t}{(q-1)^{q-1}} \right), \tag{9}$$

also given by

$$h(t) = \sum_{n=1}^{\infty} \frac{1}{(q-1)n+1} \binom{qn}{n} t^{n-1} = \sum_{n=1}^{\infty} \frac{1}{qn+1} \binom{qn+1}{n} t^{n-1}. \tag{10}$$

Moreover,

$$\big(h(t)\big)^s = {}_qF_{q-1}\left(\begin{array}{c} \frac{sq}{q}, \frac{sq+1}{q}, \ldots, \frac{(s+1)q-1}{q} \\ \frac{sq+1}{q-1}, \frac{sq+2}{q-1}, \ldots, \frac{(s+1)q-1}{q-1} \end{array} \middle| \frac{q^q t}{(q-1)^{q-1}} \right) \tag{11}$$

for every $s \in \mathbb{R}$. Consequently,

$$\big(h(t)\big)^s = \sum_{n=0}^{\infty} \frac{qs}{(q-1)n+qs} \binom{q(n+s)-1}{n} t^n$$

and, therefore,

$$[t^j]\big(th(t)\big)^s = [t^{j-s}]\big(h(t)\big)^s = \frac{qs}{(q-1)j+s} \binom{qj-1}{j-s}. \tag{12}$$

Proof. If $A(t) = (1+t)^q$, by (8), Theorem 1.3,

$$[t^{n-1}]h(t) = \frac{1}{n}[t^{n-1}](1+t)^{qn} = \frac{1}{n}\binom{qn}{n-1} = \frac{(qn)!}{((q-1)n+1)!\, n!}.$$

Therefore,

$$h(t) = \sum_{n=1}^{\infty} \frac{1}{(q-1)n+1} \binom{qn}{n} t^{n-1} = \sum_{n=0}^{\infty} \frac{1}{(q-1)n+q} \binom{qn+q}{n+1} t^n. \tag{13}$$

The series (13) is hypergeometric as its constant term is 1 and, setting $A_n = \frac{(qn+q)!}{(n+1)!((q-1)n+q)!} t^n$, the ratio of consecutive terms is

$$\begin{aligned} \frac{A_{n+1}}{A_n} &= \frac{(qn+q+1)(qn+q+2)\cdots(qn+2q)}{\big((q-1)n+q+1\big)\big((q-1)n+q+2)\big)\cdots\big((q-1)n+2q-1\big)} \cdot \frac{t}{n+2} \\ &= \frac{\big(n+\frac{q+1}{q}\big)\big(n+\frac{q+2}{q}\big)\cdots\big(n+\frac{2q}{q}\big)}{\big(n+\frac{q+1}{q-1}\big)\big(n+\frac{q+2}{q-1}\big)\cdots\big(n+\frac{2q-1}{q-1}\big)} \cdot \frac{n+1}{n+2} \cdot \frac{t}{n+1} \cdot \frac{q^q}{(q-1)^{q-1}} \\ &= \frac{\big(n+\frac{q}{q}\big)\big(n+\frac{q+1}{q}\big)\cdots\big(n+\frac{2q-1}{q}\big)}{\big(n+\frac{q+1}{q-1}\big)\big(n+\frac{q+2}{q-1}\big)\cdots\big(n+\frac{2q-1}{q-1}\big)} \cdot \frac{1}{n+1} \cdot \frac{q^q t}{(q-1)^{q-1}}. \end{aligned}$$

Thus, (9) follows immediately.

Note that $h(t) = \frac{\mathcal{B}_q(t)-1}{t}$, where $\mathcal{B}_q$ is the generalized binomial series, given by

$$\mathcal{B}_q(t) = \sum_{n=0}^{\infty} \frac{1}{qn+1}\binom{qn+1}{n} t^n,$$

for which we have

$$\big(\mathcal{B}_q(t)\big)^r = \sum_{n=0}^{\infty} \frac{r}{qn+r}\binom{qn+r}{n} t^n, \tag{14}$$

for any real r (see [6], page 201, and also [14], where in Theorem 2.1 this is obtained from Lagrange's Inversion Theorem, using (3)). From (14), by the same argument used for the series (13), it follows that, for any $r \in \mathbb{R}$,

$$\big(\mathcal{B}_q(t)\big)^r = {}_qF_{q-1}\left(\begin{array}{c} \frac{r}{q}, \frac{r+1}{q}, \ldots, \frac{r+q-1}{q} \\ \frac{r+1}{q-1}, \frac{r+2}{q-1}, \ldots, \frac{r+q-1}{q-1} \end{array} \middle| \, \frac{q^q t}{(q-1)^{q-1}} \right). \tag{15}$$

On the other hand, since $h(t) = A\big(th(t)\big) = \big(1 + th(t)\big)^q = \big(\mathcal{B}_q(t)\big)^q$, replacing r by qs in (15), we have (11) for any $s \in \mathbb{R}$ and, therefore, (12) holds. $\square$

Remark. From (15), we obtain the remarkable identity

$$\begin{aligned} &\left[{}_qF_{q-1}\left(\begin{array}{c} \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q}{q} \\ \frac{2}{q-1}, \frac{3}{q-1}, \ldots, \frac{q}{q-1} \end{array} \middle| \, \frac{q^q t}{(q-1)^{q-1}} \right) \right]^r \\ &\qquad = {}_qF_{q-1}\left(\begin{array}{c} \frac{r}{q}, \frac{r+1}{q}, \ldots, \frac{r+q-1}{q} \\ \frac{r+1}{q-1}, \frac{r+2}{q-1}, \ldots, \frac{r+q-1}{q-1} \end{array} \middle| \, \frac{q^q t}{(q-1)^{q-1}} \right) \end{aligned} \tag{16}$$

for powers of a hypergeometric function, which is essentially contained in (5.60) of [6] but is not explicitly stated in the literature. Indeed, the literature does not refer to many instances in which a product of hypergeometric functions is also hypergeometric. In Section 5 we obtain two more identities involving a product of hypergeometric functions.

3 The Identities of Andrews

We now apply the procedure described in Theorem 2.1 of extracting new Riordan arrays from a given one to provide a new proof of some identities obtained by G. E. Andrews in [1], namely

$$F_n = \sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{\lfloor \frac{1}{2}(n-1-5k) \rfloor} \tag{17}$$

and

$$F_n = \sum_{k=-\infty}^{\infty} (-1)^k \binom{n}{\lfloor \frac{1}{2}(n-1-5k) \rfloor}, \tag{18}$$

where (F_n) is the sequence of Fibonacci numbers, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Different proofs of (17) and (18) were given by H. Gupta in [7] and by M. D. Hirschhorn in [9] and [10]. They are all rather involved, though elementary, and they are specifically designed to deal with the case of Pascal's triangle. As indicated in [7] and [9], identities (17) and (18) are equivalent to

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1} \right], \tag{19}$$

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+2}{n-5j} - \binom{2n+2}{n-5j-1} \right] \tag{20}$$

and

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2} \right], \tag{21}$$

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{2n}{n-5j} - \binom{2n}{n-5j-2} \right], \tag{22}$$

respectively. In [2] G. E. Andrews proves these identities in the context of identities of the Rogers–Ramanujan type (see also [11]). In [3], identities (19) through (22), as well as several other similar identities for trinomial coefficients and Catalan's triangle, have been proved in a very elementary and direct way.

We now prove (20) to illustrate how identities (19) through (22) can be obtained by a Riordan array technique. Replacing n by $n-1$ in (20), it suffices to show that

$$F_{2n} = \sum_{j=-\infty}^{\infty} \left[\binom{2n}{n-5j-1} - \binom{2n}{n-5j-2} \right]. \tag{23}$$

We start with a visualization of Pascal's triangle in which alternate rows have been removed and only non-vanishing binomial numbers are represented:

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & \mathbf{+1} & 2 & 1 & & & & & \\
 & & & & \mathbf{-1} & \mathbf{+4} & 6 & 4 & 1 & & & & \\
 & & & 1 & \mathbf{-6} & \mathbf{+15} & 20 & 15 & 6 & \mathbf{-1} & & & \\
 & & 1 & 8 & \mathbf{-28} & \mathbf{+56} & 70 & 56 & 28 & \mathbf{-8} & \mathbf{+1} & & \\
 & 1 & 10 & 45 & \mathbf{-120} & \mathbf{+210} & 252 & 210 & 120 & \mathbf{-45} & \mathbf{+10} & 1 & \\
\mathbf{+1} & 12 & 66 & 220 & \mathbf{-495} & \mathbf{+792} & 924 & 792 & 495 & \mathbf{-220} & \mathbf{+66} & 12 & 1 \\
 & & & & & & \cdots\cdots & & & & & &
\end{array}$$

Identity (23) corresponds to adding in each row the elements marked with a plus sign and subtracting the ones marked with a minus. By symmetry, we can represent this sum using only the right-hand side of the above table, which by Theorem 2.1 is the following Riordan array $\tilde{d}_{n,k} = \binom{2n}{n+k}$ with marked plus and minus entries

$$\begin{array}{ccccccc}
1 & & & & & & \\
2 & \mathbf{+1} & & & & & \\
6 & \mathbf{+4} & \mathbf{-1} & & & & \\
20 & \mathbf{+15} & \mathbf{-6} & \mathbf{-1} & & & \\
70 & \mathbf{+56} & \mathbf{-28} & \mathbf{-8} & \mathbf{+1} & & \\
252 & \mathbf{+210} & \mathbf{-120} & \mathbf{-45} & \mathbf{+10} & 1 & \\
924 & \mathbf{+792} & \mathbf{-495} & \mathbf{-220} & \mathbf{+66} & 12 & \mathbf{+1} \\
 & & & \cdots\cdots & & &
\end{array}$$

In order to prove (23), we wish to evaluate the sum

$$S_n = \sum_{j=-\infty}^{\infty} \left[\binom{2n}{n-5j-1} - \binom{2n}{n-5j-2} \right],$$

i. e.,

$$S_n = \sum_{k=0}^{\infty} \left[\binom{2n}{n+5k+1} - \binom{2n}{n+5k+2} - \binom{2n}{n+5k+3} + \binom{2n}{n+5k+4} \right].$$

In terms of the Riordan array $\tilde{d}_{n,k} = \binom{2n}{n+k}$, $n, k \geq 0$, we have

$$S_n = \sum_{k=0}^{\infty} f_k \tilde{d}_{n,k}\,, \tag{24}$$

where

$$f(t) = \sum f_k t^k = t - t^2 - t^3 + t^4 + t^6 - t^7 - t^8 + t^9 + \cdots = \frac{t - t^2 - t^3 + t^4}{1 - t^5}\,.$$

For Pascal's triangle the generating function of the $A-$sequence is $1+t$, since $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. Hence, by Theorem 2.1, the generating function of the $A-$sequence for $\{\tilde{d}_{n,k}\}$ is

$$A(t) = (1+t)^2. \tag{25}$$

Either using Theorem 2.2 or noting that it follows from (7) that $h(t)$ satisfies

$$t^2h^2 + (2t-1)h + 1 = 0, \tag{26}$$

we obtain $h(t) = \frac{1-2t-\sqrt{1-4t}}{2t^2}$. Therefore, $\{\tilde{d}_{n,k}\}$ is the Riordan array characterized by the pair $\big(d(t), h(t)\big)$, where

$$d(t) = \sum_{n=0}^{\infty} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}$$

and

$$h(t) = \frac{1 - 2t - \sqrt{1-4t}}{2t^2}. \tag{27}$$

By Theorem 1.2, S_n given by (24) satisfies $S_n = [t^n]\, d(t) f(th(t))$. Note that

$$f(t) = \frac{t-t^2-t^3+t^4}{1-t^5} = \frac{t(1-t)(1-t^2)}{(1-t)(1+t+t^2+t^3+t^4)} = \frac{t^{-1} - t}{t^{-2} + t^{-1} + 1 + t + t^2}\,.$$

Setting $w := th(t)$, by (26) w and w^{-1} are the roots of the equation

$$y^2 + \frac{2t-1}{t} y + 1 = 0.$$

Hence,

$$w^{-1} + w = \frac{1-2t}{t}, \qquad w^{-1} - w = \frac{\sqrt{1-4t}}{t}$$

and

$$w^2 + w^{-2} = (w + w^{-1})^2 - 2 = \frac{1 - 4t + 2t^2}{t^2}.$$

Therefore,

$$f(th(t)) = f(w) = \frac{\frac{\sqrt{1-4t}}{t}}{\frac{1-4t+2t^2}{t^2} + \frac{1-2t}{t} + 1} = \frac{t\sqrt{1-4t}}{1-3t+t^2}$$

and, finally,

$$d(t)f\big(th(t)\big) = \frac{t}{1-3t+t^2}. \tag{28}$$

It is well known that both sequences of Fibonacci numbers F_{2n} and F_{2n+1} satisfy the recurrence relation $x_n = 3x_{n-1} - x_{n-2}$ and have generating functions

$$\frac{t}{1-3t+t^2} = \sum_{n=0}^{\infty} F_{2n}t^n$$

and

$$\frac{1-t}{1-3t+t^2} = \sum_{n=0}^{\infty} F_{2n+1}t^n,$$

respectively (see [12], page 230). Hence, equation (28) implies that

$$d(t)f\big(th(t)\big) = \sum_{n=0}^{\infty} F_{2n}t^n,$$

from which (23) and (20) follow.

Identities (19), (21), and (22) can be obtained in a similar way. For (19) and (21) we eliminate the even-numbered rows of Pascal's triangle

$$\begin{array}{ccccc|ccccc}
 & & & & 1 & 1 & & & & \\
 & & & 1 & 3 & 3 & 1 & & & \\
 & & 1 & 5 & 10 & 10 & 5 & 1 & & \\
 & \cdot & \cdot & \cdot & 35 & 35 & 21 & 7 & 1 & \\
\cdot & \cdot & \cdot & \cdot & \cdot & 126 & 84 & 36 & 9 & 1 \\
 & & & & \multicolumn{2}{c}{\cdots\cdots} & & & &
\end{array}$$

and consider the right-hand side of what remains, obtaining by Theorem 2.1 a Riordan array for which

$$d(t) = \sum_{n=0}^{\infty} \binom{2n+1}{n+1} t^n = \frac{1}{2t}\left(\frac{1}{\sqrt{1-4t}} - 1\right)$$

and

$$h(t) = \frac{1-2t-\sqrt{1-4t}}{2t^2}.$$

4 Further Identities

In this section, to illustrate the usefulness of the construction considered in Theorem 2.1, we apply it to obtain a few more identities via our Riordan array approach. Some of these identities are well-known, while others are not. We believe that identities (37) and (39) are new. We need one more property of Riordan arrays, which generalizes a well-known property of Pascal's triangle.

Theorem 4.1 *If $D = \big(d(t), h(t)\big)$ is a Riordan array, then for any integers $k \geq s \geq 1$ we have*

$$d_{n,k} = \sum_{j=s}^{n} d_{n-j,k-s}[t^j]\big(th(t)\big)^s. \tag{29}$$

Proof. From (4), it follows that $d_{n,k} = [t^n]\, d(t)\big(th(t)\big)^{k-s}\big(th(t)\big)^s$. Hence,

$$d_{n,k} = \sum_{j=s}^{n} \Big([t^{n-j}]\, d(t)\big(th(t)\big)^{k-s}\Big)\Big([t^j]\big(th(t)\big)^s\Big),$$

i.e.,

$$d_{n,k} = \sum_{j=s}^{n} d_{n-j,k-s}[t^j]\big(th(t)\big)^s. \qquad \square$$

One particular case of (29), for $s = 1$, is

$$d_{n,k} = \sum_{j=1}^{n} h_{j-1} d_{n-j,k-1}, \tag{30}$$

where $h(t) = \sum h_k t^k$. For example, in the case of Pascal's triangle, since $h_k = 1$ for all k, we obtain the well-known identity

$$\sum_{j=1}^{n} \binom{n-j}{k-1} = \binom{n}{k}. \tag{31}$$

More generally, for Pascal's triangle, we have

$$[t^j]\big(th(t)\big)^s = [t^j]\frac{t^s}{(1-t)^s} = [t^{j-s}](1-t)^{-s} = \binom{-s}{j-s}(-1)^{j-1} = \binom{j-1}{s-1}$$

and, therefore, (29) becomes

$$\sum_{j=s}^{n} \binom{n-j}{k-s}\binom{j-1}{s-1} = \binom{n}{k},$$

which is (5.26) of [6], page 169, and contains (31) as a particular case.

We now apply formulas (29) or (30) to Riordan arrays obtained by the method described in Theorem 2.1.

Example 4.1 For fixed integers $p \geq 2$ and $r \geq 0$, starting with Pascal's triangle and deleting $p-1$ rows after each line kept, as described in Theorem 2.1, we obtain the Riordan array

$$d_{n,k} = \binom{pn+r}{(p-1)n+r+k} = \binom{pn+r}{n-k}.$$

Note that $d_{n,0} = \binom{pn+r}{(p-1)n+r} = \binom{pn+r}{n}$ and

$$d(t) = \sum_{n=0}^{\infty} \binom{pn+r}{n} t^n = \sum_{n=0}^{\infty} \frac{(pn+r)!}{n!((p-1)n+r)!} t^n. \tag{32}$$

This is a hypergeometric series. By the same argument used for the series (13), we get

$$d(t) = {}_pF_{p-1}\left(\begin{array}{c} \frac{r+1}{p}, \frac{r+2}{p}, \ldots, \frac{r+p}{p} \\ \frac{r+1}{p-1}, \frac{r+2}{p-1}, \ldots, \frac{r+p-1}{p-1} \end{array} \middle| \, \frac{p^p t}{(p-1)^{p-1}} \right).$$

For $p = 2$ and $r = 0, 1$, another expression in closed form for (32) is

$$d(t) = \sum_{n=0}^{\infty} \binom{2n+r}{n} t^n = \frac{1}{\sqrt{1-4t}} \left(\frac{1-\sqrt{1-4t}}{2t} \right)^r$$

(see [14], Corollary 2.2).

In this example, the function h is given by (9) or, equivalently, by (10) with $q = p$. Combining (29) and (12) it follows that

$$\sum_{j=s}^{n} \frac{ps}{(p-1)j+s} \binom{pj-1}{j-s} \binom{p(n-j)+r}{n-j-k+s} = \binom{pn+r}{n-k}. \tag{33}$$

In the particular case $s = 1$, (33) becomes

$$\sum_{j=1}^{n} \frac{p}{(p-1)j+1} \binom{pj-1}{j-1} \binom{p(n-j)+r}{n-j-k+1} = \binom{pn+r}{n-k},$$

or,

$$\sum_{j=1}^{n} \frac{1}{pj+1} \binom{pj+1}{j} \binom{p(n-j)+r}{n-j-k+1} = \binom{pn+r}{n-k}$$

and, finally, adding $\binom{pn+r}{n-k+1}$ to both sides,

$$\sum_{j=0}^{n} \frac{1}{pj+1} \binom{pj+1}{j} \binom{p(n-j)+r}{n-j-k+1} = \binom{pn+r+1}{n-k+1}. \tag{34}$$

It would be nice to find a combinatorial interpretation for (34), since there are several interpretations for the generalized Catalan number $\frac{1}{pj+1}\binom{pj+1}{j} = \frac{1}{(p-1)j+1}\binom{pj}{j}$ (see [6], page 360, and [8]).

Setting $j = i + s$, $x = ps$, $y = pk - ps + r$, and replacing n by $n + k$, identity (33) becomes

$$\sum_{i=0}^{n} \frac{x}{x+pi} \binom{x+pi}{i} \binom{y+p(n-i)}{n-i} = \binom{x+y+pn}{n},$$

which is (5.62) of [6].

Example 4.2 We now consider Catalan's triangle $d_{n,k} = \dfrac{k+1}{n+1}\dbinom{2(n+1)}{n-k}$, for $n, k \geq 0$,

1						
2	1					
5	4	1				
14	14	6	1			
42	48	27	8	1		
132	165	110	44	10	1	
429	572	429	208	65	12	1
			......			

This array was introduced in [18] and has a nice interpretation in terms of pairs of paths on a lattice. On the bidimensional lattice $\mathbb{Z}^2$, consider all paths that start at the origin, consist of unit steps and are such that all steps go East or North. The length of a path is the number of steps in the path. The distance between two paths of length n with end-points (a_n, b_n) and (a'_n, b'_n), respectively, is $|a_n - a'_n|$. Two paths are said to be non-intersecting if the origin is the only point in common. Let $B(n,k)$, for $1 \le k \le n$, denote the number of pairs of non-intersecting paths of length n whose distance from one another is k. The array defined by $d_{n,k} = B(n+1, k+1)$ is called Catalan's triangle. It is shown in [18] that $B(n,k) = \frac{k}{n}\binom{2n}{n-k}$ and that

$$B(n-1,k-1) + 2B(n-1,k) + B(n-1,k+1) = B(n,k).$$

Therefore, by Theorem 1.3, $d_{n,k} = \frac{k+1}{n+1}\binom{2(n+1)}{n-k}$ is a Riordan array and (25) is the generating function of its $A-$sequence. The first column of this triangle is formed by

$$d_{n,0} = \frac{1}{n+1}\binom{2(n+1)}{n} = \frac{1}{n+2}\binom{2(n+1)}{n+1} = C_{n+1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$ is the n^{th} Catalan number. We then have

$$d(t) = h(t) = \frac{1 - 2t - \sqrt{1-4t}}{4t^2}.$$

For fixed integers $p \ge 2$ and $r \ge 0$, we consider the Riordan array

$$\tilde{d}_{n,k} = d_{pn+r,(p-1)n+k+r} = \frac{(p-1)n + r + k + 1}{pn + r + 1}\binom{2(pn+r+1)}{n-k}, \tag{35}$$

for which, by Theorem 2.1, the generating function of the $A-$sequence is $A(t) = (1+t)^{2p}$. By Theorem 2.2, it follows that for the Riordan array (35) we have

$$\big(h(t)\big)^s = {}_{2p}F_{2p-1}\left(\begin{array}{c} \frac{2ps}{2p}, \frac{2ps+1}{2p}, \ldots, \frac{2p(s+1)-1}{2p} \\ \frac{2ps+1}{2p-1}, \frac{2ps+2}{2p-1}, \ldots, \frac{2p(s+1)-1}{2p-1} \end{array} \middle| \, \frac{(2p)^{2p}t}{(2p-1)^{2p-1}}\right)$$

and

$$[t^j]\big(th(t)\big)^s = [t^{j-s}]\big(h(t)\big)^s = \frac{2ps}{(2p-1)j + s}\binom{2pj-1}{j-s}. \tag{36}$$

From (29) and (36) it follows that

$$\sum_{j=s}^{n} \frac{2ps}{(2p-1)j+s}\binom{2pj-1}{j-s}\frac{(p-1)(n-j)+r+k-s+1}{p(n-j)+r+1}\binom{2(p(n-j)+r+1)}{n-j-k+s}$$
$$= \frac{(p-1)n+r+k+1}{pn+r+1}\binom{2(pn+r+1)}{n-k}, \tag{37}$$

for every integers $s \le k \le n$, $p \ge 1$, and $r \ge 0$. Identity (37) is probably new.

Example 4.3 In [13] the following variant of Catalan's triangle arises as an example of the infinite matrix associated to a generating tree

$$d_{n,k} = \begin{cases} \dfrac{k+1}{n+1}\dbinom{2n-k}{n}, & \text{if } 0 \le k \le 2n, \\ 0, & \text{if } 0 \le 2n < k. \end{cases}$$

$$\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
2 & 2 & 1 & & & & \\
5 & 5 & 3 & 1 & & & \\
14 & 14 & 9 & 4 & 1 & & \\
42 & 42 & 28 & 14 & 5 & 1 & \\
132 & 132 & 90 & 48 & 20 & 6 & 1 \\
 & & & \cdots\cdots & & &
\end{array}$$

In this case,

$$d_{n+1,k+1} = d_{n,k} + d_{n,k+1} + d_{n,k+2} + \cdots + d_{n,n}$$

and, thus,

$$A(t) = 1 + t + t^2 + \cdots = \frac{1}{1-t}.$$

From (7) it follows that $h(t)$ satisfies $th^2 - h + 1 = 0$ and, therefore, $h(t) = \frac{1-\sqrt{1-4t}}{2t}$. But this is precisely the generating function of the Catalan numbers, which form the first column of the triangle. Hence,

$$d(t) = h(t) = \frac{1-\sqrt{1-4t}}{2t}.$$

Note that

$$\frac{1-\sqrt{1-4t}}{2t} = {}_2F_1\left(\left.\begin{array}{c}\frac{1}{2}, \frac{2}{2} \\ \frac{2}{1}\end{array}\right| 4t\right).$$

It follows from (16) that

$$\big(h(t)\big)^s = {}_2F_1\left(\left.\begin{array}{c}\frac{s}{2}, \frac{s+1}{2} \\ s+1\end{array}\right| 4t\right) = \sum_{n=0}^{\infty} \frac{s}{2n+s}\binom{2n+s}{n} t^n.$$

For fixed integers $p \geq 2$ and $r \geq 0$, we consider the Riordan array $\tilde{d}_{n,k} = d_{pn+r,(p-1)n+k+r}$. Then, for $n \geq 0$,

$$\tilde{d}_{n,k} = \frac{(p-1)n+k+r+1}{pn+r+1}\binom{(p+1)n+r-k}{pn+r}$$

if $0 \leq k \leq (p+1)n+r$, and $\tilde{d}_{n,k} = 0$ otherwise. By Theorem 2.1, the generating function of the $A-$sequence of $\{\tilde{d}_{n,k}\}$ is $\tilde{A}(t) = 1/(1-t)^p$. By (8), the $h-$function $\tilde{h}$ of $\{\tilde{d}_{n,k}\}$ satisfies

$$[t^{n-1}]h(t) = \frac{1}{n}[t^{n-1}]\big(A(t)\big)^n = \frac{1}{n}[t^{n-1}](1-t)^{-pn}$$

$$= \frac{(-1)^n}{n}\binom{-pn}{n-1} = \frac{(pn+n-2)!}{n!(pn-1)!}$$

and, therefore,

$$\tilde{h}(t) = \sum_{n=1}^{\infty} \frac{1}{(p+1)n-1}\binom{(p+1)n-1}{n} t^{n-1} = \sum_{n=0}^{\infty} \frac{\big((p+1)n+p-1\big)!}{(n+1)!(pn+p-1)!} t^n.$$

The function $\tilde{h}$ is hypergeometric. By the same argument used for the series (13), we have

$$\tilde{h}(t) = {}_{p+1}F_p\left(\left.\begin{array}{c}\frac{p}{p+1}, \frac{p+1}{p+1}, \cdots, \frac{2p}{p+1} \\ \frac{p+1}{p}, \frac{p+2}{p}, \cdots, \frac{2p}{p}\end{array}\right| \frac{(p+1)^{p+1}}{p^p} t\right).$$

It follows from (16) that

$$\tilde{h}(t) = \left[{}_{p+1}F_p \left(\begin{matrix} \frac{1}{p+1}, \frac{2}{p+1}, \dots, \frac{p+1}{p+1} \\ \frac{2}{p}, \frac{3}{p}, \dots, \frac{p+1}{p} \end{matrix} \,\middle|\, \frac{(p+1)^{p+1}}{p^p} t \right) \right]^p$$

and also

$$\big(\tilde{h}(t)\big)^s = {}_{p+1}F_p \left(\begin{matrix} \frac{sp}{p+1}, \frac{sp+1}{p+1}, \dots, \frac{sp+p}{p+1} \\ \frac{sp+1}{p}, \frac{sp+2}{p}, \dots, \frac{sp+p}{p} \end{matrix} \,\middle|\, \frac{(p+1)^{p+1}}{p^p} t \right), \tag{38}$$

for any $s \in \mathbb{R}$. It is straightforward from (38) that

$$\big(\tilde{h}(t)\big)^s = \sum_{n=0}^{\infty} \frac{sp}{(p+1)n + sp} \binom{(p+1)n + sp}{n} t^n$$

and, therefore,

$$[t^j]\big(t\tilde{h}(t)\big)^s = [t^{j-s}]\big(\tilde{h}(t)\big)^s = \frac{sp}{(p+1)j - s} \binom{(p+1)j - s}{j - s}.$$

Thus, by Theorem 4.1,

$$\begin{aligned} \sum_{j=s}^{n-k+s} \frac{ps}{(p+1)j-s} \binom{(p+1)j-s}{j-s} \frac{(p-1)(n-j)+k-s+r+1}{p(n-j)+r+1} \binom{(p+1)(n-j)+r-k+s}{p(n-j)+r} \\ = \frac{(p-1)n + k + r + 1}{pn + r + 1} \binom{(p+1)n + r - k}{pn + r} \end{aligned} \tag{39}$$

for $s \le k \le n$. Setting $j = i + s$, $x = ps$, $y = pk - ps + r$, and replacing n by $n + k$, identity (39) can be rewritten as

$$\begin{aligned} \sum_{i=0}^{n} \frac{x}{(p+1)i+x} \binom{(p+1)i+x}{i} \frac{(p-1)(n-i)+y+1}{p(n-i)+y+1} \binom{(p+1)(n-i)+y}{n-i} \\ = \frac{(p-1)n + x + y + 1}{pn + x + y + 1} \binom{(p+1)n + x + y}{n}. \end{aligned} \tag{40}$$

Note that, for fixed $n \ge k \ge s \ge 0$ and $r \ge 0$ integers, our argument only proves (40) for special values of x and y, namely of the form $x = ps$ and $y = p(k - s) + r$, with $p \ge 2$ an integer. As usual, this is enough to guarantee that (40) holds for all x and y real, since both sides of (40) are polynomials in p.

Identity (40) seems to be new, though it resembles the old formula

$$\sum_{i=0}^{n} \frac{x}{zi+x} \binom{zi+x}{i} \frac{y}{z(n-i)+y} \binom{z(n-i)+y}{n-i} = \frac{x+y}{zn+x+y} \binom{zn+x+y}{n} \tag{41}$$

due to Rothe (1793) and Hagen (1891) (see (3.142), in Gould's collection [5], or (5.63) in, [6]).

5 Final comments

Often identities of the same type as the ones obtained in Section 4 can be proved directly employing generating functions, Lagrange's Inversion Formula, and standard Riordan array techniques. Indeed, in [21] Sprugnoli provides this kind of proof to most of the identities appearing in Gould's large collection [5]. As an example, we give a direct proof of (40) along these lines. First note that

$$\frac{(p-1)n+y+1}{pn+y+1}\binom{(p+1)n+y}{n} = \left(1-\frac{n}{pn+y+1}\right)\binom{(p+1)n+y}{n}$$

$$= \binom{(p+1)n+y}{n} - \binom{(p+1)n+y}{n-1} = [t^n](1+t)^{(p+1)n+y} - [t^{n-1}](1+t)^{(p+1)n+y}$$

$$= [t^n](1-t)(1+t)^y(1+t)^{(p+1)n}.$$

On the other hand, by Lagrange's Inversion Formula (3),

$$\mathcal{G}\Big([t^n](1-t)(1+t)^y\big((1+t)^{p+1}\big)^n\Big) = \left[\frac{(1-w)(1+w)^y}{1-t(p+1)(1+w)^p}\,\middle|\, w=t(1+w)^{p+1}\right]$$

$$= \left[\frac{(1-w)(1+w)^{y+1}}{1-pw}\,\middle|\, w=t(1+w)^{p+1}\right].$$

Hence,

$$\begin{aligned}\mathcal{A}(p,y;t) &:= \sum_{n=0}^{\infty}\frac{(p-1)n+y+1}{pn+y+1}\binom{(p+1)n+y}{n}t^n \\ &= \left[\frac{(1-w)(1+w)^{y+1}}{1-pw}\middle|w=t(1+w)^{p+1}\right].\end{aligned} \tag{42}$$

By Theorem 2.1 of [14], we have the following identity for the function given by (14)

$$\begin{aligned}\big(\mathcal{B}_{p+1}(t)\big)^x &= \sum_{n=0}^{\infty}\frac{x}{(p+1)n+x}\binom{(p+1)n+x}{n}t^n \\ &= \big[(1+w)^x\big|w=t(1+w)^{p+1}\big].\end{aligned} \tag{43}$$

Combining (42) and (43) yields

$$\big(\mathcal{B}_{p+1}(t)\big)^x\cdot\mathcal{A}(p,y;t) = \mathcal{A}(p,x+y;t) \tag{44}$$

and, therefore, applying $[t^n]$ to both sides, (40) holds. Identity of Rothe–Hagen (41) mentioned above follows from the application of $[t^n]$ to both sides of the equality $\big(\mathcal{B}_p(t)\big)^{x+y} = \big(\mathcal{B}_p(t)\big)^x\cdot\big(\mathcal{B}_p(t)\big)^y$. It is interesting to observe that (44) implies that $\mathcal{A}(p,y;t)$ is a function of exponential type on y, as the ratio $\mathcal{A}(p,x+y;t)/\mathcal{A}(p,y;t)$ does not depend on y. As a matter of fact, by (42) and (43),

$$\mathcal{A}(p,y;t) = \frac{(2-\mathcal{B}_{p+1}(t))\mathcal{B}_{p+1}(t)}{1+p-p\mathcal{B}_{p+1}(t)}\big(\mathcal{B}_{p+1}(t)\big)^y.$$

We can restate (44) in terms of hypergeometric functions. If we consider the general term

$$A_n = \frac{(p-1)n+y+1}{pn+y+1}\binom{(p+1)n+y}{n}t^n$$

of the power series in (42) and calculate the ratio of consecutive terms, we find

$$\frac{A_{n+1}}{A_n}=\frac{\left(n+\frac{y+1}{p+1}\right)\left(n+\frac{y+2}{p+1}\right)\cdots\left(n+\frac{y+p+1}{p+1}\right)}{\left(n+\frac{y+2}{p}\right)\left(n+\frac{y+3}{p}\right)\cdots\left(n+\frac{y+p+1}{p}\right)}\cdot\frac{n+\frac{y+p}{p-1}}{n+\frac{y+1}{p-1}}\cdot\frac{1}{n+1}\cdot\frac{(p+1)^{p+1}t}{p^p},$$

from which it follows that

$$\mathcal{A}(p,y;t)={}_{p+2}F_{p+1}\left(\begin{matrix}\frac{y+1}{p+1},\frac{y+2}{p+1},\dots,\frac{y+p+1}{p+1},\frac{y+p}{p-1}\\ \frac{y+2}{p},\frac{y+3}{p},\dots,\frac{y+p+1}{p},\frac{y+1}{p-1}\end{matrix}\;\middle|\;\frac{(p+1)^{p+1}t}{p^p}\right).$$

Finally, identity (44) can be stated in terms of hypergeometric functions as

$$\begin{aligned}&{}_{p+1}F_{p}\left(\begin{matrix}\frac{x}{p+1},\frac{x+1}{p+1},\dots,\frac{x+p}{p+1}\\ \frac{x+1}{p},\frac{x+2}{p},\dots,\frac{x+p}{p}\end{matrix}\;\middle|\;\frac{(p+1)^{p+1}t}{p^p}\right)\cdot{}_{p+2}F_{p+1}\left(\begin{matrix}\frac{y+1}{p+1},\frac{y+2}{p+1},\dots,\frac{y+p+1}{p+1},\frac{y+p}{p-1}\\ \frac{y+2}{p},\frac{y+3}{p},\dots,\frac{y+p+1}{p},\frac{y+1}{p-1}\end{matrix}\;\middle|\;\frac{(p+1)^{p+1}t}{p^p}\right)\\ &\quad={}_{p+2}F_{p+1}\left(\begin{matrix}\frac{x+y+1}{p+1},\frac{x+y+2}{p+1},\dots,\frac{x+y+p+1}{p+1},\frac{x+y+p}{p-1}\\ \frac{x+y+2}{p},\frac{x+y+3}{p},\dots,\frac{x+y+p+1}{p},\frac{x+y+1}{p-1}\end{matrix}\;\middle|\;\frac{(p+1)^{p+1}t}{p^p}\right).\end{aligned}$$

Of course this identity looks simpler if we further replace $(p+1)^{p+1}t/p^p$ by t, but then the coefficients of the corresponding developments in power series are no longer integers.

By a similar argument, (37) can be rephrased as

$$\begin{aligned}\sum_{i=0}^{n}\frac{2x}{(2p-1)i+2x}\binom{2pi+2x-1}{i}\frac{(p-1)(n-i)+y+1}{p(n-i)+y+1}\binom{2\big(p(n-i)+y+1\big)}{n-i}\\ =\frac{(p-1)n+x+y+1}{pn+x+y+1}\binom{2(pn+y+1)}{n}.\end{aligned}\tag{45}$$

Also as above, we find

$$\begin{aligned}\mathcal{C}(p,x;t)&:=\sum_{n=0}^{\infty}\frac{2x}{(2p-1)n+2x}\binom{2pn+2x-1}{n}t^n\\ &=\left[(1+w)^{2x}\;\middle|\;w=t(1+w)^{2p}\right]\\ \mathcal{D}(p,y;t)&:=\sum_{n=0}^{\infty}\frac{(p-1)n+y+1}{pn+y+1}\binom{2(pn+y+1)}{n}t^n\\ &=\left[\frac{(1-w)(1+w)^{2y+2}}{1+(1-2p)w}\;\middle|\;w=t(1+w)^{2p}\right].\end{aligned}$$

Hence, $\mathcal{C}(p,x;t)=\big(\phi(t)\big)^x$ and $\mathcal{D}(p,y;t)=\psi(t)\big(\phi(t)\big)^y$, where $\phi(t)=\mathcal{C}(p,1;t)$ and $\psi(t)=(2-\mathcal{C}(p,\frac{1}{2};t))\mathcal{C}(p,1;t)/(2p+(1-2p)\mathcal{C}(p,\frac{1}{2};t))$. Therefore,

$$\mathcal{C}(p,x;t)\cdot\mathcal{D}(p,y;t)=\mathcal{D}(p,x+y;t).\tag{46}$$

Applying $[t^n]$ to both sides, (45) follows. Using the same argument as above, we can show that

$$\mathcal{C}(p,x;t)={}_{2p}F_{2p-1}\left(\begin{matrix}\frac{2x}{2p},\frac{2x+1}{2p},\dots,\frac{2x+2p-1}{2p}\\ \frac{2x+1}{2p-1},\frac{2x+2}{2p-1},\dots,\frac{2x+2p-1}{2p-1}\end{matrix}\;\middle|\;\frac{(2p)^{2p}\,t}{(2p-1)^{2p-1}}\right)=\big(\mathcal{B}_{2p}(t)\big)^{2x}$$

and

$$\mathcal{D}(p,y;t) = {}_{2p+2}F_{2p+1}\left(\begin{matrix} \frac{2y+3}{2p}, \frac{2y+4}{2p}, \ldots, \frac{2y+2p+2}{2p}, \frac{y+p}{p-1}, \frac{y+1}{p} \\ \frac{2y+3}{2p-1}, \frac{2y+4}{2p-1}, \ldots, \frac{2y+2p+2}{2p-1}, \frac{y+1}{p-1}, \frac{y+p+1}{p} \end{matrix} \,\middle|\, \frac{(2p)^{2p}\,t}{(2p-1)^{2p-1}} \right).$$

Hence, by (46), another identity involving a product of hypergeometric functions can be derived.

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