GENERALIZATION OF AN IDENTITY OF ANDREWS

Eduardo H. M. Brietzke
Instituto de Matemática – UFRGS
Caixa Postal 15080
91509–900 Porto Alegre, RS, Brazil
email: brietzke@mat.ufrgs.br

Abstract

We consider an identity relating Fibonacci numbers to Pascal's triangle discovered by G. E. Andrews. Several authors provided proofs of this identity, all of them rather involved or else relying on sophisticated number theoretical arguments. We not only give a simple and elementary proof, but also show the identity generalizes to arrays other than Pascal's triangle. As an application we obtain identities relating trinomial coefficients and Catalan's triangle to Fibonacci numbers.

There is a well-known identity relating the sequence of Fibonacci numbers to Pascal's triangle. Not so well-known are identities

$$F_n = \sum_{k=-\infty}^{\infty} (-1)^k \binom{n-1}{\left\lfloor \frac{1}{2}(n-1-5k) \right\rfloor} \tag{1}$$

and

$$F_n = \sum_{k=-\infty}^{\infty} (-1)^k \binom{n}{\left\lfloor \frac{1}{2}(n-1-5k) \right\rfloor}, \tag{2}$$

obtained by G. E. Andrews in [1]. Different proofs of (1) and (2) were given by H. Gupta in [4] and by M. D. Hirschhorn in [5] and [6]. They are all specifically designed to deal with the case of Pascal's triangle. As indicated in [4] and [5], identities (1) and (2) are equivalent to

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-1} \right], \tag{3}$$

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+2}{n-5j} - \binom{2n+2}{n-5j-1} \right] \tag{4}$$

and

$$F_{2n+2} = \sum_{j=-\infty}^{\infty} \left[\binom{2n+1}{n-5j} - \binom{2n+1}{n-5j-2} \right], \tag{5}$$

$$F_{2n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{2n}{n-5j} - \binom{2n}{n-5j-2} \right], \tag{6}$$

respectively. These identities have been reobtained by G. E. Andrews [2] in the context of identities of the Rogers–Ramanujan type (see also [7]).

The purpose of this article is to provide a simple and entirely elementary proof for identities (3) through (6) as well as several other similar identities. Our method has the advantage of showing that this kind of identity holds not only for Pascal's triangle, but also for any other array constructed in a similar way. As an application of our result, we obtain versions of these identities for the trinomial coefficients [the identities (9) and (10) below], as well as for Catalan's triangle [(11) and (12)].

We start with a visualization of Pascal's triangle in which alternate rows have been removed and only nonvanishing binomial numbers are represented:

$$\begin{array}{cccccccccc}
 & & & & \boxed{1} & 1 & & & & \\
 & & & \enclose{circle}{1} & \boxed{3} & 3 & 1 & & & \\
 & & 1 & \enclose{circle}{5} & \boxed{10} & 10 & 5 & 1 & & \\
 & 1 & 7 & \enclose{circle}{21} & \boxed{35} & 35 & 21 & 7 & \enclose{circle}{1} & \\
1 & 9 & 36 & \enclose{circle}{84} & \boxed{126} & 126 & 84 & 36 & \enclose{circle}{9} & \boxed{1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \tag{7}$$

Identity (3) is visualised considering in each row the sum of the elements represented inside rectangles and subtracting from it the sum of elements represented inside circles. The remaining identities can be likewise visualized. Note that the above array consists of a first row in which the only nonvanishing elements are two 1's followed by rows in which each element is obtained by adding up the element above it to the right, the element above it to the left, and 2 times the element directly above it. In general we have the following theorem.

Theorem. *Let* $s(0,k)$*, with* $k \in \mathbb{Z}$*, be an arbitrary sequence such that* $s(0,k) \neq 0$ *for only finitely many values of* k*. Given* $\alpha, \beta \in \mathbb{R}$*, define* $s(n,k)$ *recursively, for* $n \in \mathbb{N}$ *and* $k \in \mathbb{Z}$*, setting*

$$s(n,k) = \alpha s(n-1,k-1) + \beta s(n-1,k) + \alpha s(n-1,k+1).$$

Then for any fixed k_0 *the sequence* $\big(d_n\big)$ *defined by*

$$d_n = \sum_{j=-\infty}^{\infty} \Big[s(n, k_0 - 5j) - s(n, k_0 - 5j - 1)\Big]$$

satisfies the recurrence relation

$$d_n = \big(2\beta - \alpha\big)d_{n-1} + \big(\alpha\beta + \alpha^2 - \beta^2\big)d_{n-2}\,.$$

Proof: If suffices to prove the relation for $n = 2$, since the general case follows from it considering the row $\big(s(n-2,k)\big)_k$ as being the first.

For each fixed n, the n^{th} row of the array depends linearly on the first. In addition, the operator $\mathcal{L}$ defined by

$$\mathcal{L}(a_n) = \sum_{j=-\infty}^{\infty} \Big[a_{k_1-5j} - a_{k_1-5j-1}\Big]$$

maps two-tailed sequences linearly to real numbers.

It therefore suffices to prove the proposition in the particular case where the first row contains an element, say $s(0,0)$, equal to 1 and all others equal to 0. Now it is a simple matter of inspecting the table

0	0	0	1	0	0	0
0	0	α	β	α	0	0
0	α^2	$2\alpha\beta$	$2\alpha^2+\beta^2$	$2\alpha\beta$	α^2	0

Since the null sequence satisfies any homogeneous linear recurrence relation, it is enough to check that $(d_0, d_1, d_2) = \pm(0, \alpha, 2\alpha\beta - \alpha^2)$ and $(d_0, d_1, d_2) = \pm(1, \beta - \alpha, 2\alpha^2 + \beta^2 - 2\alpha\beta)$ satisfy the relation $d_2 = \big(2\beta - \alpha\big)d_1 + \big(\alpha\beta + \alpha^2 - \beta^2\big)d_0$. This completes the proof.

Corollary. *Under the same hypotheses as in the theorem, for any given* $k_0 \in \mathbb{Z}$ *and* $k_1 \in \mathbb{N}$*, the sequence* $\big(d_n\big)$ *defined by*

$$d_n = \sum_{j=-\infty}^{\infty} \Big[s(n, k_0 - 5j) - s(n, k_0 - 5j - k_1)\Big]$$

satisfies the recurrence relation

$$d_n = \big(2\beta - \alpha\big)d_{n-1} + \big(\alpha\beta + \alpha^2 - \beta^2\big)d_{n-2}\,.$$

Proof: The proof follows by linearity, writing out

$$d_n = d_n^{(1)} + \cdots + d_n^{(k_1)}\,,$$

with

$$d_n^{(i)} = \sum_{j=-\infty}^{\infty} \Big[s(n, k_0 - i + 1 - 5j) - s(n, k_0 - i - 5j)\Big],$$

and applying the theorem.

Application 1. Identities (3) through (6) hold. We first apply the theorem to the array (7), in which $s(n,k) = \binom{2n+1}{k+n}$, for $-n \le k \le n+1$, and 0 otherwise. Since $\alpha = 1$ and $\beta = 2$, it follows that the right-hand side of (3) defines a sequence satisfying the recurrence relation $d_n = 3d_{n-1} - d_{n-2}$. But it is a well-known fact that this recurrence relation is also satisfied by both sequences $d_n = F_{2n+1}$ and $d_n = F_{2n+2}$. Therefore, it suffices to verify (3) for n equal to 0 and 1. Identities (4) through (6) can be obtained by the same argument, applied to (7) or to the array obtained by deleting the even-numbered rows of Pascal's triangle.

Application 2. The Fibonacci numbers appear in a similar manner when we operate with the array of the trinomial coefficients. The trinomial coefficients $\binom{n}{k}_2$, for $|k| \le n$, are defined by (see [3], section 6.2)

$$\big(1 + x + x^2\big)^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^{n+k}$$

and satisfy

$$\binom{n}{k}_2 = \sum_j (-1)^j \binom{n}{j}\binom{2n-2j}{n-j-k}.$$

Using the property

$$\binom{n}{k}_2 = \binom{n-1}{k-1}_2 + \binom{n-1}{k}_2 + \binom{n-1}{k+1}_2, \tag{8}$$

we construct the array of the trinomial coefficients, as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & & \boxed{1} & & & & & & \\
 & & & & & \textcircled{1} & \boxed{1} & 1 & & & & & \\
 & & & & 1 & \textcircled{2} & \boxed{3} & 2 & 1 & & & & \\
 & & & 1 & 3 & \textcircled{6} & \boxed{7} & 6 & 3 & 1 & & & \\
 & & 1 & 4 & 10 & \textcircled{16} & \boxed{19} & 16 & 10 & 4 & \textcircled{1} & & \\
 & \boxed{1} & 5 & 15 & 30 & \textcircled{45} & \boxed{51} & 45 & 30 & 15 & \textcircled{5} & \boxed{1} & \\
. & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

We claim that the trinomial coefficients satisfy the identities

$$F_n = \sum_{j=-\infty}^{\infty} \left[\binom{n+1}{5j}_2 - \binom{n+1}{5j-1}_2 \right] \tag{9}$$

and

$$F_{n+1} = \sum_{j=-\infty}^{\infty} \left[\binom{n}{5j}_2 - \binom{n}{5j-2}_2 \right]. \tag{10}$$

For example, identity (9) corresponds to adding in each row the elements inside rectangles and subtracting the ones inside circles in the above representation.

Since $\alpha = \beta = 1$ by (8), our theorem and its corollary imply that the right-hand sides of identities (9) and (10) define sequences satisfying the recurrence relation $d_n = d_{n-1} + d_{n-2}$. Hence, the proof of these identities consists in verifying them for $n = 0$ and 1.

Application 3. On the bidimensional lattice $\mathbb{Z}^2$, consider all paths that start at the origin, consist of unit steps and are such that all steps go East or North. The length of a path is the number of steps in the path. The distance between two paths of length n with end-points (a_n, b_n) and (a'_n, b'_n), respectively, is $|a_n - a'_n|$. Two paths are said to be non-intersecting if the origin is the only point in common. Let $B(n,k)$, for $1 \le k \le n$, denote the number of pairs of non-intersecting paths of length n whose distance from one another is k. The array defined by the $B(n,k)$ is called Catalan's triangle (see [8]) and its first column is formed by Catalan numbers $C_n = B(n,1) = \frac{1}{n+1}\binom{2n}{n}$, which, among other things, count rooted binary trees and the lattice paths from $(0,0)$ to (n,n) that stay above the main diagonal. Catalan's triangle satisfies the same recurrence relation

$$B(n,k) = B(n-1,k-1) + 2B(n-1,k) + B(n-1,k+1)$$

as alternate rows in Pascal's triangle, for $k \geq 2$, while for $k = 1$ we have

$$B(n,1) = 2B(n-1,1) + B(n-1,2).$$

In order to fit Catalan's triangle into our framework, we embed it into the larger array in which the first row is $s(1,\pm 1) = \pm 1$, $s(1,k) = 0$, if $k \neq \pm 1$, and the subsequent rows are given by

$$s(n,k) = s(n-1,k-1) + 2s(n-1,k) + s(n-1,k+1), \quad (n \geq 2,\ k \in \mathbb{Z})$$

as follows:

$$\begin{array}{ccccccccccccc}
 & & & & & -1 & 0 & 1 & & & & & \\
 & & & & -1 & -2 & 0 & 2 & 1 & & & & \\
 & & & -1 & -4 & -5 & 0 & 5 & 4 & 1 & & & \\
 & & -1 & -6 & -14 & -14 & 0 & 14 & 14 & 6 & 1 & & \\
 & -1 & -8 & -27 & -48 & -42 & 0 & 42 & 48 & 27 & 8 & 1 & \\
-1 & -10 & -44 & -110 & -165 & -132 & 0 & 132 & 165 & 110 & 44 & 10 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

The portion of the array lying to the the right of the zero column is Catalan's triangle, i.e., $s(n,k) = B(n,k)$, if $1 \leq k \leq n$.

By the above theorem and corollary, several identities follow. However, because $k \mapsto s(n,k)$ is odd, some of these identities vanish trivially. We point out two nontrivial identities. Since

$$\sum_{j=-\infty}^{\infty} \big[s(n,5j+1) - s(n,5j+4)\big] = 2\sum_{j=0}^{\infty}\big[B(n,5j+1) - B(n,5j+4)\big],$$

using the same argument as before, we can show the identity

$$F_{2n-1} = \sum_{j=0}^{\infty}\big[B(n,5j+1) - B(n,5j+4)\big]. \tag{11}$$

Likewise one may show that

$$F_{2n-2} = \sum_{j=0}^{\infty} \big[B(n, 5j+2) - B(n, 5j+3) \big]. \tag{12}$$

According to [8], Proposition 2.1, $B(n,k) = \frac{k}{n}\binom{2n}{n-k}$. Substituting in (13) and (14), yields

$$F_{2n-1} = \sum_{j=0}^{\infty} \left[\frac{5j+1}{n} \binom{2n}{n-5j-1} - \frac{5j+4}{n} \binom{2n}{n-5j-4} \right] \tag{13}$$

and

$$F_{2n-2} = \sum_{j=0}^{\infty} \left[\frac{5j+2}{n} \binom{2n}{n-5j-2} - \frac{5j+3}{n} \binom{2n}{n-5j-3} \right]. \tag{14}$$

Formulas (13) and (14) above appear to be new.

Acknowledgment. The author would like to express his gratitude to Prof. J. P. O. Santos for bringing identities (3) and (5) to his attention. He is also grateful to Prof. J. Cigler for pointing out that a $q-$version of (1) and (2) already appeared in the work [9] by I. Schur from 1917.

References

[1] G. E. Andrews, Some formulae for the Fibonacci sequence with generalizations, *Fibonacci Quart.*, **7** (1969), 113–130.

[2] G. E. Andrews, A polynomial identity which implies the Rogers–Ramanujan identities, *Scripta Math.*, **28** (1970), 297–305.

[3] F. R. Bernhart, Catalan, Motzkin, and Riordan numbers, *Discrete Math.*, **204** (1999), 73–112.

[4] H. Gupta, The Andrews formula for Fibonacci numbers, *Fibonacci Quart.*, **16** (1978), 552–555.

[5] M. D. Hirschhorn, The Andrews formula for Fibonacci numbers, *Fibonacci Quart.*, **19** (1981), 1–2.

[6] M. D. Hirschhorn, Solution to problem 1621, *Mathematics Magazine*, **75** (2002), 149–250.

[7] M. Ivković and J. P. O. Santos, Fibonacci numbers and partitions, *Fibonacci Quart.*, **41** (2003), 263–278.

[8] L. W. Shapiro, A Catalan triangle, *Discrete Math.*, **14** (1976), 83–90.

[9] I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche. In: I. Schur, Gesammelte Abhandlungen. Vol. 2, pp. 118–136. Springer, Berlin 1973.

AMS Classification Numbers: 11B39, 05A19